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Auger Electrons Resulting from *K*-Capture

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The most intense group of Auger electrons due to x-rays associated with *K*-capture in Zn^{65} would have energies lying in a narrow band within about 300 ev of a maximum value of 7056 ev ($H\rho 284$). These would arise from $K-L^2$ transitions in the resulting Cu atom. Electrons from the $K-LM$ transitions would be about $\frac{1}{4}$ as numerous, lying within about 300 ev of a maximum value of 8016 ev ($H\rho 302$). The soft electrons which are emitted copiously from Zn^{65} were found to have an absorption limit in collodion of $0.15\pm.03$ mg/cm², approximately 7500 ev. When samples were placed 0.6 to 1.3 cm from the screen of a screen wall-counter filled with H_2 at 2 cm Hg pressure, the upper limit (by inspection) was found to be at $H\rho 302\pm 30$ (8000 ± 1700 ev). Scattering in the counter gas and absorption in the sample are believed negligible. These data are consistent with the hypothesis that the soft electrons from Zn^{65} are Auger electrons resulting from *K*-capture. Similar results, in less detail, were obtained in deflection experiments with Fe^{55}.

WHEN an unstable isotope decays by *K*-capture, the resulting vacancy in the *K*-shell may be filled by an *L*, *M*, ··· electron, and the characteristic x-rays of the product atom are emitted.[1] Indeed, one strong proof for *K*-capture has been identification of the atomic number of the atom emitting the x-rays, since if emitted as a result of *K*-capture the energy levels of the *product* atom are involved. It is not necessary, however, that all *K*-capture processes result in emission of an x-ray quantum. By the Auger effect, a radiationless transfer may take place, the single vacancy in the *K*-shell being replaced by two vacancies in the *L*-shell ($K-L^2$ transition), or by a vacancy in the *L* shell and one in the *M* shell ($K-LM$ transition). The energies of the Auger electrons emitted in such transitions depend entirely upon the atomic number of the product atom resulting from *K*-capture.

[1] L. W. Alvarez, Phys. Rev. **54**, 486 (1938).

EXPERIMENTS WITH Zn^{65}

The 250-day isotope Zn^{65} decays by *K*-capture,[1] as well as by emission of positrons of maximum energy[2,3] 0.4 to 0.7 Mev. A γ-ray of 1.0 Mev is present, but the absence of many fast electrons shows it to be only slightly converted,[4] a result to be expected from the low value of Z^2. Compton recoils from the γ-ray are also observed.[5] Apparently *K*-capture is favored as an alternative process to positron emission, since there is an anomalously high γ/β-ratio, and annihilation radiation is present to an inappreciable extent.[4] Most of the Cu nuclei must be formed in an excited state, giving rise to the γ-ray. The Cu $K\alpha$

[2] L. DuBridge *et al.*, quoted in J. J. Livingood and G. T. Seaborg, Rev. Mod. Phys. **12**, 30 (1940).
[3] S. W. Barnes and G. Valley, Phys. Rev. **53**, 946(A), (1938).
[4] J. J. Livingood and G. T. Seaborg, Phys. Rev. **55**, 457 (1939).
[5] R. Sagane, Phys. Rev. **55**, 31 (1939).

x-radiation has been found[1] by means of differential absorption measurements.

Calculated Auger Energies

The energies of the Auger electrons from the resulting Cu atom were calculated to the required

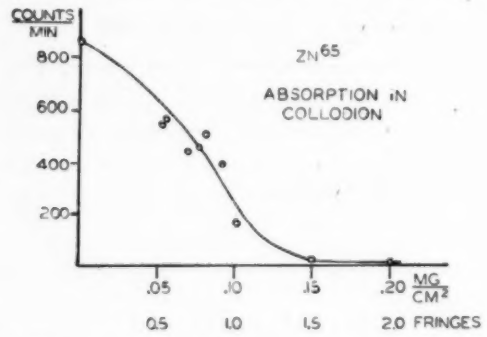

FIG. 1. Absorption in collodion of soft electrons from Zn^{65}.

degree of accuracy by the method of Ference,[6] using Slater's[7] screening constant theory. Term values were taken from Siegbahn.[8] The most energetic $K-L^2$ Auger electrons have an energy given approximately by $E_K-2E_{L_{III}}=8954-2(933)=7088$ ev. When account is taken of the extra work required to remove the second L_{III} electron due to changes in screening, a correction factor $\Delta E=2E_2-E_1-E_3=32$ ev must be subtracted, leaving 7056 ev ($H\rho 284$ gauss-cm). If we assume extreme jj coupling, we find the lower limit for $K-L^2$ will be $E_K-2E_{L_I}-\Delta E=6730$ ev ($H\rho 277$), but judging from Ference's experimental work with germanium, these are less numerous. Deviations from jj coupling will determine the detailed structure of the Auger levels, but will not appreciably affect the extreme values here listed. Thus we conclude that the $K-L^2$ electrons are a nearly homogeneous group, lying within about 300 ev of a maximum value of 7056 ev.

Similarly, the $K-LM$ group, which is expected[6] to be about $\frac{1}{3}$ as numerous as the $K-L^2$ group, lies within about 300 ev of a maximum value of $E_K-(E_{L_{III}}+E_{M_{IV,V}})=8016$ ev ($H\rho 302$). The screening correction in this case is negligible.

Direct measurements by Robinson and Cassie[9] on the Auger electrons resulting from photoionization of Cu by Mo $K\alpha\beta$ gave lines between $H\rho 280$ and $H\rho 292$, consistent with the theoretical values derived above.

Apparatus

A screen wall-counter[10] was constructed having a counter wall 1.9 cm in diameter and 10 cm long. The wall was of copper or galvanized iron window screen, the insulation was of Lucite, and the wire was No. 30 gauge bare copper. The sample could be placed 1.3 cm from the screen, or closer if desired. Decay constants were checked by means of a thin glass-walled argon-oxygen counter. A Neher-Harper[11] quenching circuit and a scale-of-16 scaling circuit[12] were used to record pulses. A cathode-ray oscilloscope was always used to check the behavior of the screen wall counter, which had a cosmic-ray background of about 40 per minute.

Many very soft electrons were found to be emitted by invisibly thin zinc samples electroplated on copper or aluminum foils. A sample giving about one count per minute (above a background of 8) on the glass counter gave about 500 per minute in the screen wall-counter. The Zn^{65} was over 2 years old at the time of the experiments (1943), and the decay curve, obtained from γ's registered on the glass counter, was straight over two half-lives, showing a slope of 258 ± 10 days.

Absorption Experiments

Thin films of collodion placed 1 mm from the source were used in absorption experiments in the screen wall-counter. The chief error lay in determination of the thicknesses of the films, which were estimated in two ways: (1) from the

TABLE I. Absorption in collodion of soft electrons from Zn^{65}.

Method	Cut-off
Color thickness	0.13 mg/cm²
Fringes	0.15
Weighing	0.2

[6] M. Ference, Phys. Rev. **51**, 720 (1937).
[7] J. C. Slater, Phys. Rev. **36**, 57 (1930).
[8] M. Siegbahn, *Spectroskopie der Röntgenstrahlen* (Verlagsbuchhandlung Julius Springer, Berlin, 1931).
[9] H. Robinson and A. M. Cassie, Proc. Roy. Soc. **A113**, 282 (1928).
[10] W. F. Libby and D. D. Lee, Phys. Rev. **55**, 245 (1939).
[11] H. V. Neher and W. W. Harper, Phys. Rev. **49**, 940 (1936).
[12] H. Lifschutz and J. L. Lawson, Rev. Sci. Inst. **9**, 83 (1938).

interference colors[13] and (2) from the shift of fringes of Na light in a Michelson interferometer. Reasonable agreement was obtained. An index of refraction of 1.5 and a specific gravity of 1.66 were taken for collodion. A very rough check was made with a microbalance. The shape of the absorption curve is shown in Fig. 1; the points at 1.0, 1.5, and 2.0 fringes were determined by the interferometer and that at 1.0 fringe was also determined from the film's color so as to provide overlapping of the two methods. Cut-off was at about $0.15\pm.03$ mg/cm^2 (see Table I). Extrapolating Schonland's[14] data for homogeneous cathode rays, an $H\rho$ value of 285 was found (7500 ev).

Magnetic Deflection Experiments

The screen wall-counter was placed in a longitudinal magnetic field formed by an electromagnet from which pole pieces of diameter 10 cm had been removed. The coils were each 9 cm long and 34 cm in outside diameter, and their centers were 20 cm apart. A rough approximation to Helmholtz coils was thus obtained. The counter was placed at the center of the system, and the magnetic field was explored by a search coil and found to be uniform to within 0.2 percent perpendicular to the axis, and to within 3 percent along the length of the counter. At cut-off a much shorter portion of the axis was used, indicating a more uniform field at cut-off than the outside limits of 3 percent. The constant of the magnet was determined as 99 ± 3 gauss/amp. with the aid of a search coil and a ballistic galvanometer. As the magnet current was increased, eventually only those electrons were counted which left the source in a direction perpendicular both to the field and to the line joining the source to the counter. Thus the radius of curvature was half the closest distance of source to screen. A typical curve is shown in Fig. 2. It will be noted, as found by Libby and Lee,[10] that a sharper cut-off was obtained with H_2 gas at 2 cm of Hg than with argon at 1 cm, because of scattering in the counter gas. In each case a trace of alcohol vapor was added to improve the counting characteristics. It is believed that errors due to scattering in the gas and self-absorption in the source were negligibly small. The weighted mean of four runs, made at values of ρ from 0.27 to 0.64 cm, gave an end point (by inspection) of 302 ± 30 gauss-cm, or $E_{max}=8000\pm1700$ ev. The rather large error arises from the difficulty of estimating end points (±5 percent), lack of homogeneity in the magnetic field (±3 percent), and uncertainty in ρ (±5 percent). The sign of the particles was determined as negative, using sloping shields in the counter.

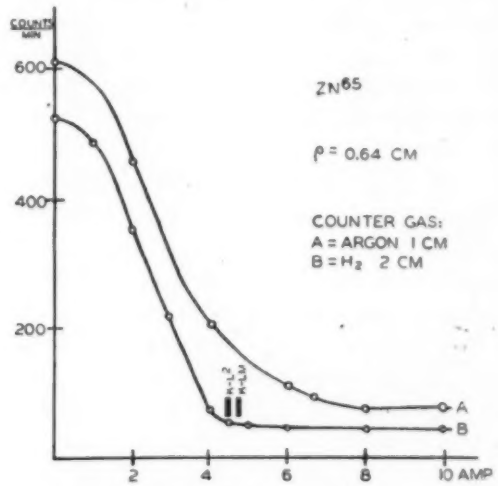

FIG. 2. Magnetic deflection of soft electrons from Zn^{65}, using screen wall-counter. A sharper cut-off is obtained when H_2 is used as the counter gas. Calculated Auger electron energies are indicated, those arising from the $K-LM$ transitions being less numerous than those arising from the $K-L^2$ transitions.

From the shape of Curve B in Fig. 2, it would seem that the screen wall-counter can be used successfully at least down to 8000 ev electrons, which is considerably lower than the lowest energies measured by Libby and Lee.

Discussion of Results

Within the experimental errors, it is seen that both the absorption and the deflection experiments are consistent with the interpretation of these electrons as Auger electrons resulting from K-capture (See Table II).

Very few of the many experiments involving K-capture have been performed under circumstances allowing detection of the extremely low energy Auger electrons. Plesset[15] looked for

[13] O. D. Chwolson, *Traité de Physique* (Libraire Scientifique A. Hermann, Paris, 1906), Vol. 2, p. 596.
[14] B. F. J. Schonland, Proc. Roy. Soc. **A108**, 187 (1925).
[15] E. H. Plesset, Phys. Rev. **62**, 181 (1942).

TABLE II. Maximum Auger energies for Zn^{65}.

Method	Maximum energy (ev)	
Theory	7056	$(K-L^2)$
	8016	$(K-LM)$
Absorption	7500±1500	
Deflection	8150±1700	

electrons from Zn^{65} in the range 20 to 300 kev and found none. Valley and McCreary[16] searched for Auger electrons from K-capture in the 6.4 hr. $Cd^{107,109}$, and concluded that electrons of energies <20 kev have little action on the photographic plate. Hemmendinger[17] looked for x-rays from Mn^{52}. He found a soft radiation which he was unable to identify, apparently because of absorption in air. Unless the source was placed inside his ionization chamber, this radiation would be considerably harder than the Auger electrons. Hurst and Pool[18] have found "electrons of very low energy associated with the fluorescence yield" in the K-capture of Ag^{106}. No energy measurements were made, however.

Auger electrons would arise following emission of any strongly converted nuclear γ-ray. The justification for describing the Auger electrons of this paper as "resulting from K-capture" lies in the very small[4,15] internal conversion of the strong[4] 1.0 Mev γ-ray of Zn^{65}. Practically the only way in which an electron leaves the K shell of Zn^{65} is through capture.

[16] G. E. Valley and R. L. McCreary, Phys. Rev. **56**, 863 (1939).
[17] A. Hemmendinger, Phys. Rev. **58**, 929 (1940).
[18] L. K. Hurst and M. L. Pool, Phys. Rev. **65**, 60(A) (1944).

EXPERIMENTS WITH Fe^{55}

A small amount of Fe was available. This sample had been precipitated as Fe_3PO_4, and small traces of Mn and Co might have been present. At the time of the measurements, the sample was at least 4 years old, and showed a slow decay period of several years. Many soft electrons from this material were found in the screen counter, presumably from K-capture in Fe^{55} (4 year half-life). A sample of purified Fe only a few months old also gave numerous soft electrons when a thin source was electroplated on a copper foil. A main group of maximum $H\rho 220\pm30$ (4300±1500 ev) was found, and traces of another group up to 20,000 ev possibly were also present. These results are based upon only one run. The calculated Auger maximum energies, however, are 5179 ev $(K-L^2)$, and 5872 ev $(K-LM)$, which are not discordant with the measurements.

ACKNOWLEDGMENTS

In conclusion, the author wishes to thank Dr. Martin Kamen of the Radiation Laboratory at Berkeley, California, who supplied the Zn and the Fe_3PO_4 samples, and Dr. R. Dubach of the Washington University Medical School, who supplied the purified Fe sample.

Radioactive Scandium. II

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(Received March 12, 1945)

Sc^{47}: This radioactive isotope has been reported to have a half-life of 2.62 days and to emit 1.1 Mev electrons. These observations are not confirmed. A new radioactive isotope has, however, been produced by bombarding calcium with alpha-particles and to some extent by bombarding calcium with deuteron and proton. It emits beta-rays of 0.46 Mev and has a half-life of 3.4 days. The assignment is made to Sc^{47}.

Sc^{48}: This isotope has been produced by the reactions $Ca^{48}(p, n)$, $Ca^{48}(d, 2n)$, $Ti^{48}(n, p)$, and $V^{51}(n, \alpha)$. From the $V^{51}(n, \alpha)$ reaction gamma-rays of 1.33 Mev and beta-rays of 0.57 Mev were observed and from the $Ca^{48}(p, n)$ reaction gamma-rays of 1.35 Mev and beta-rays of 0.57 Mev were observed. Relative saturation intensities have been calculated for Sc^{43}, Sc^{44}, Sc^{47}, and Sc^{48} when produced by proton, deuteron, alpha-particle, and fast neutron bombardments. Sc^{48} was produced free of Sc^{44} by the $V^{51}(n, \alpha)$ and the $Ca^{48}(p, n)$ reactions. Electrons of 0.57 Mev and gamma-rays of 1.33 Mev were observed. Sc^{48} emits approximately 14 gamma-rays per electron. All reactions associated with the scandium region are schematically summarized in a nuclear transmutation chart.

Sc^{47}

A RADIOACTIVE isotope of scandium has been reported which decays with a half-life of 63 ± 2 hours.[1] Beta-rays of 1.1 Mev were observed. Gamma-rays were detected, but it was not certain that the 63-hour period emitted them. This period was tentatively assigned to Sc^{47}.

In the present investigation, although several attempts were made to produce this isotope, it was not observed.

A new radioactive isotope, however, has been produced by bombardments of calcium with alpha-particles. Various target holders made of copper, platinum, or gold were used in order to eliminate possible target-holder impurities in the scandium fraction.

Figure 1 shows the results obtained for a short bombardment (4 hours) and a long bombardment (2 days); the half-life period is 3.40 days.

Electrons of maximum energy 0.46 Mev were measured by absorption in aluminum. One of the absorption curves is shown in Fig. 2. Gamma-rays, if present, were too weak to measure.

The assignment of this period to Sc^{47} has been made for the following reasons: This radioactive isotope is most easily produced by bombarding calcium with alpha-particles. Since Ca^{44} is the second most abundant stable isotope of calcium, it is probably the one from which the new isotope is produced. Neither alpha-particle bombardments of potassium nor fast neutron bombardments of scandium produced the 3.4-day period. Therefore, an assignment to Sc^{42} or Sc^{44} is improbable. The 3.4-day period could not be expected at Sc^{45} or any scandium isotope of lighter mass, since no positrons nor gamma-rays were observed. Sc^{46} is eliminated by the fact that $Sc^{45}(d, p)$ and $Sc^{45}(n, \gamma)$ reactions failed to produce the isotope. A $V^{51}(n, \alpha)$ reaction produced only one period, the 1.83-day period which is obviously assigned at Sc^{48}.

The 3.4-day period has been also produced, although weakly, by bombarding calcium with protons and with deuterons.

The following reactions have, therefore, been established:

$$Ca^{44}+\alpha\rightarrow Sc^{47}+p$$
$$Ca^{46}+d\rightarrow Sc^{47}+n$$
$$Ca^{46}+p\rightarrow Sc^{47}+\gamma.$$

Sc^{48}

A half-life period of 1.83 days was first assigned to Sc^{48} by Walke.[2] Later by bombarding titanium and vanadium metals with fast neutrons he was able to make absorption measurements to deter-

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[1] H. Walke, Phys. Rev. **57**, 163 (1940).

[2] H. Walke, Phys. Rev. **52**, 777 (1937).

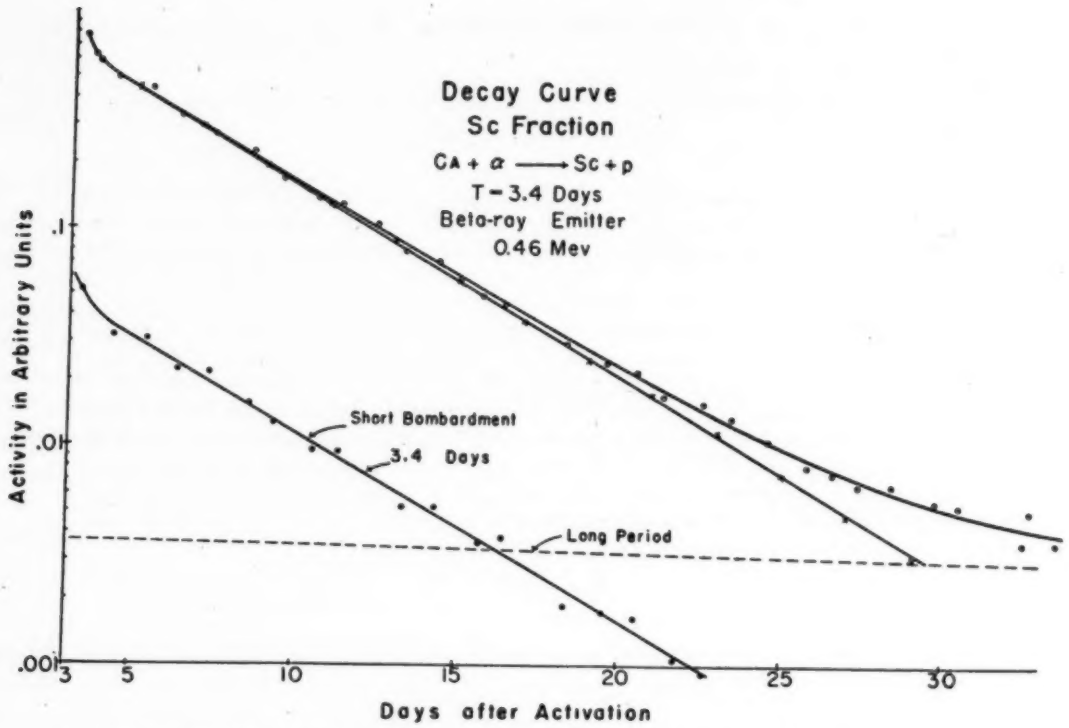

FIG. 1. Sc^{47}: Decay curve showing a half-life of 3.4 days for beta-rays.

mine the energies of the beta-rays (electrons) and of the gamma-rays. Two groups of beta-rays, one of 0.5 Mev (90 percent) and another of 1.4 Mev

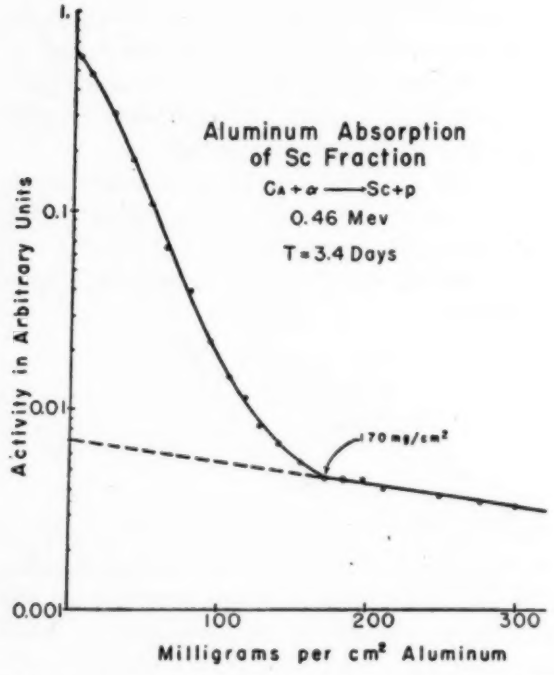

FIG. 2. Sc^{47}: Aluminum absorption measurements of the beta-ray spectrum obtained from the Ca(α, p) reaction. The end-point is 0.46 Mev.

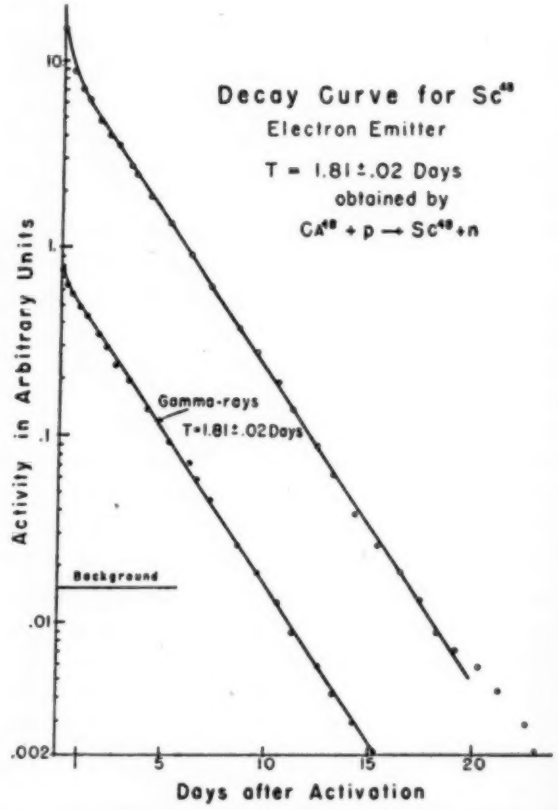

FIG. 3. Sc^{48}: Decay curve showing a half-life of 1.81 days for both beta-rays and gamma-rays obtained by the Ca(p, n) reaction.

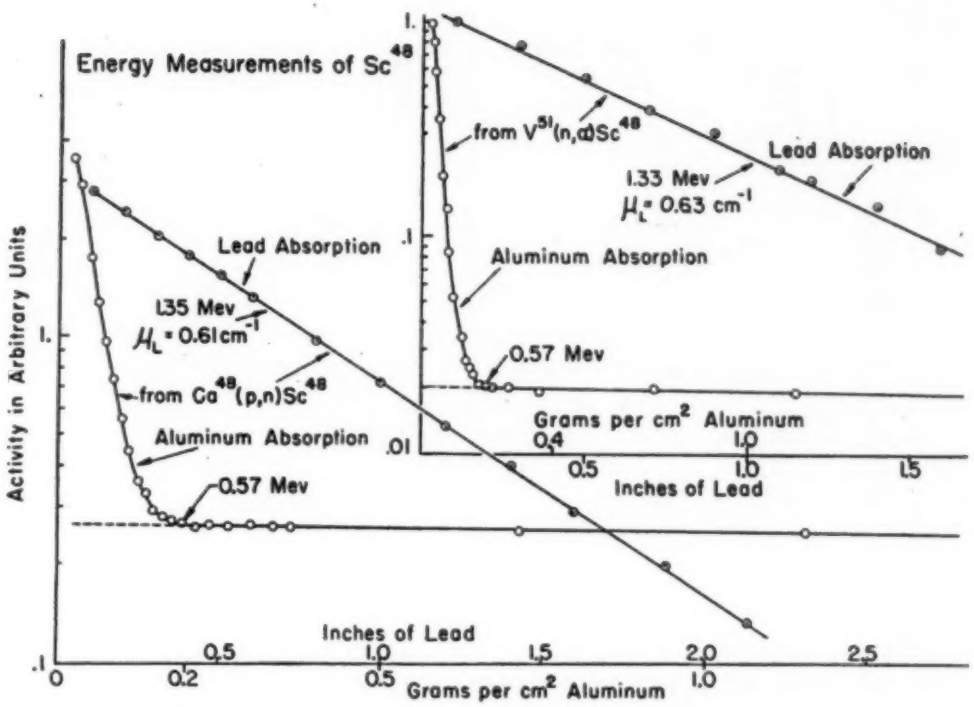

FIG. 4. Sc^{48}: Aluminum and lead absorption measurements showing a beta-ray end-point of 0.57 Mev, and a gamma-ray energy of 1.35 Mev.

(10 percent), were reported. A gamma-ray of 0.9 Mev energy was also reported.

In the present work, long bombardments with fast neutrons were made. In order to increase the amount of vanadium in the target material, vanadium pentoxide was pressed into small pellets under great pressure. In this way sufficient activity was obtained to follow the decay for ten half-lives from which the half-life was found to be 1.83 days for both gamma-rays and electrons.

By lead absorption measurements, a gamma-ray of 1.33 Mev was observed, which is the same energy as was found for the 3.92-hour period of Sc^{44}. An aluminum absorption measurement shows that only one group of electrons of maximum energy of 0.57 Mev is present.

Having determined the characteristics of Sc^{48} by the $V^{51}(n, \alpha)$ reaction, which produced Sc^{48} free of Sc^{44}, it was then found that Sc^{48} is also produced free of Sc^{44} by bombarding calcium with 5 Mev protons as shown in Fig. 3. For both the gamma-rays and electrons a half-life of 1.81 ± 0.02 days was calculated. Absorption measurements with aluminum indicate electrons of the same energy as for $V^{51}(n, \alpha)$ reactions. By absorption in lead the energy of the gamma-rays was found to be 1.35 Mev. Absorption measurements for both bombardments are shown in Fig. 4.

By the method of semicircular focusing of Compton recoils in a magnetic spectrograph Mandeville has likewise found gamma-rays which he attributed to decay of Sc^{48}. His measurements indicate a quantum energy of 1.35 Mev.[3]

Electrons of 0.64-Mev energy attributed to Sc^{48} have been reported by Smith.[4] These measurements were made with a magnetic beta-ray spectrometer on scandium fractions from calcium bombarded with deuterons.

Fast neutron bombardments of titanium produce weak Sc^{48} activities.

In all bombardments, the ratio of the intensity of the ionization produced by the beta-rays to that by the gamma-rays is approximately 15, which corresponds to an emission of about 14 gamma-rays per electron.

K-electron capture is therefore strongly suggested in lieu of many excited titanium nuclear

[3] C. E. Mandeville, Phys. Rev. **62**, 555 (1942); **64**, 147 (1943).
[4] Gail P. Smith, Phys. Rev. **61**, 578 (1942).

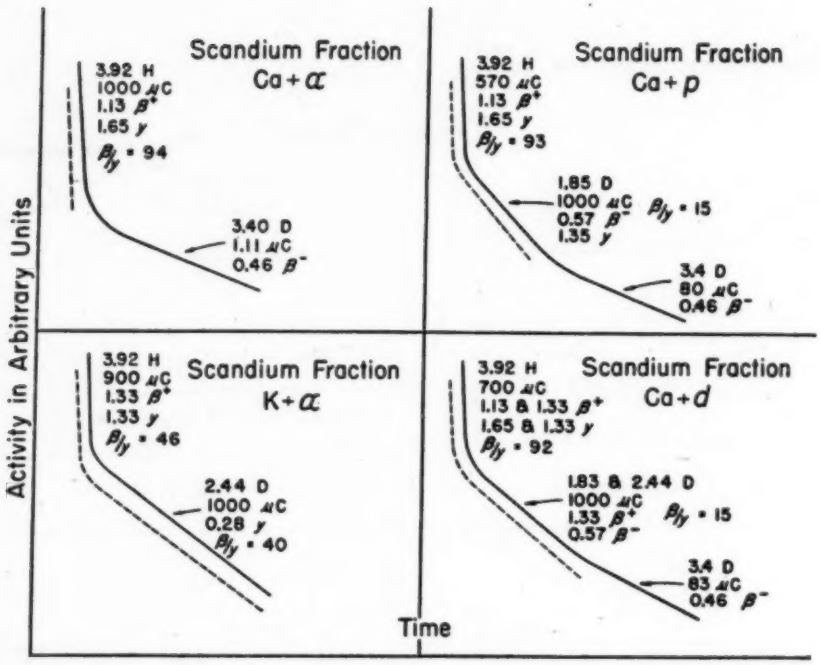

FIG. 5. Schematic summary of relative saturation intensities of various scandium periods obtained by various reactions.

levels.[5] This suggestion was given support by several absorption measurements which were made with the inverted electrometer and the thinnest pieces of aluminum and beryllium obtainable. Characteristic x-rays associated with K-electron capture for this region are of the order of 3A. The experimental values obtained ranged between 2.4A and approximately 3A.

RELATIVE INTENSITIES

All of the radioactive periods and their relative intensities are schematically shown in Fig. 5. Solid lines represent typical decay curves for beta-rays and broken lines, gamma-rays.

Relative intensities, as calculated for infinite bombardment, are indicated on the arbitrary basis of 1000 for the most intense half-life period produced. The ratio of the intensity of the beta-rays to the intensity of the gamma-rays is indicated for each half-life period.

Two main periods were produced by alpha-particle bombardments of calcium. The 3.4-day period of Sc^{47} and the 3.92-hour period of Sc^{43} have saturation intensities of 1.11 and 1000, respectively. The latter is the most intense period of scandium and may have an initial intensity 30,000 times background.

The isomers of Sc^{44}, when produced by the $K^{41}(\alpha, n)$ reaction, are found to have approximately equal saturation intensities; namely 900 for the 3.92-hour period and 1000 for the 2.44-day period.

Sc^{48} was the main isotope produced by 5 Mev proton bombardments of calcium. Sc^{43} was also

TABLE I. Summary of results.

Active isotope	Observed periods	Produced by	Energy of beta-rays in Mev	Energy of gamma-rays in Mev
Sc^{42}	Not observed			
Sc^{43}	3.92 hours	Ca-α-p Ca-p-n Ca-d-n Ca-d-$2n$	1.13(+)	1.65
Sc^{44}	3.92 hours	K-α-n Sc-n-$2n$ Ca-d-n Ca-d-$2n$	1.33(+)	1.33
	2.44 days			0.28
Sc^{47}	3.40 days	Ca-α-p Ca-d-n Ca-p-γ	0.46(−)	
Sc^{48}	1.83 days	V-n-α Ti-n-p Ca-d-$2n$ Ca-p-n	0.57(−)	1.33

[5] Ernest Pollard, Phys. Rev. **54**, 411 (1938).

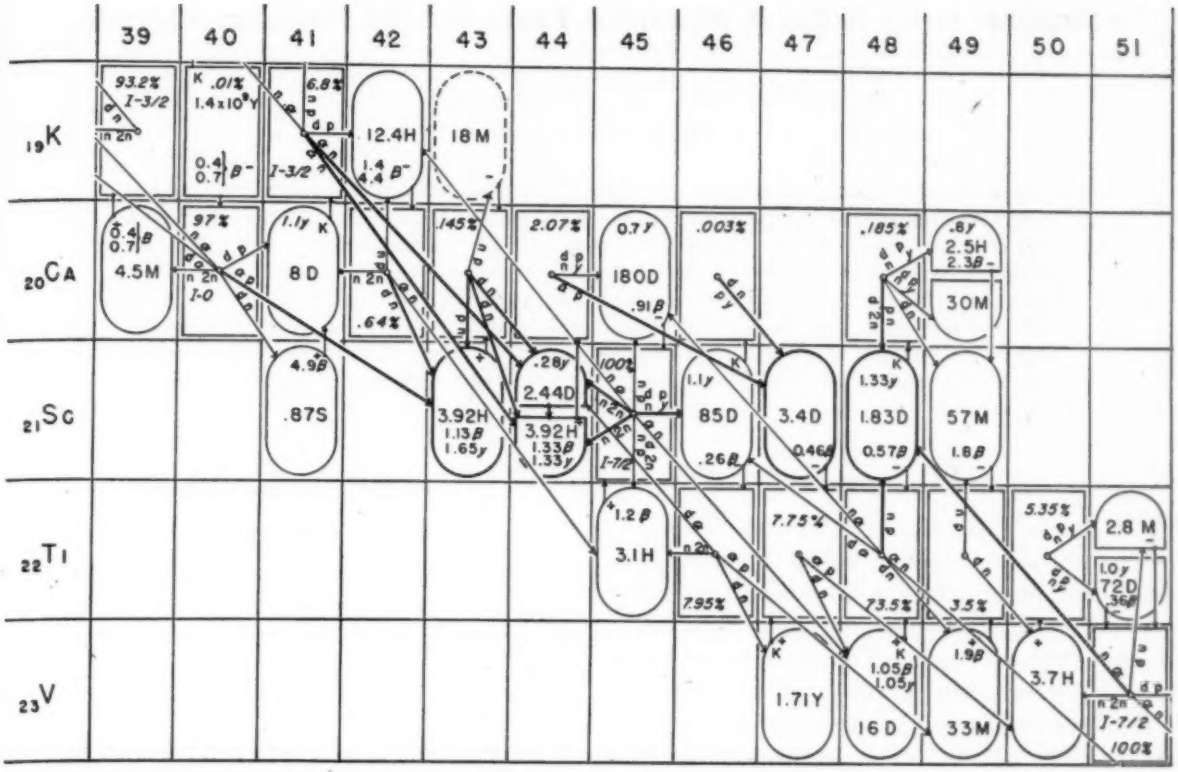

FIG. 6. Transmutation chart showing the scandium region and reactions leading to radioactive scandium isotopes.

produced, but no evidence was found to indicate Sc^{44}; however, a weak period of 3.4 days due to Sc^{47} was present. The saturation intensities of these isotopes, Sc^{48}, Sc^{43}, Sc^{44}, and Sc^{47}, are, respectively, 1000, 570, 0, and 80.

Deuteron-bombardments of calcium produced several radioactive isotopes of scandium, namely Sc^{43}, Sc^{44}, Sc^{47}, and Sc^{48}. Sc^{48} was the main isotope produced and was approximately seven times as intense as Sc^{44}. The composite decay curve of these activities is so complex that an attempt to discover the individual periods by subtractions could only fail. The half-life discrepancies reported in the literature based upon deuteron bombardments may in this way be reconciled.

The saturation intensity of the Sc^{43} 3.92-hour period plus the Sc^{44} 3.92-hour period is 700. The intensities of the 1.83-day period of Sc^{48} plus the 2.44 day period of Sc^{44} is 1000. The 3.4-day period of Sc^{47} has an intensity of 83.

The results of this investigation and that reported earlier[6] are summarized in Table I and Fig. 6.

ACKNOWLEDGMENT

It is a pleasure to express gratitude for support received from the University Development Fund and from Mr. Julius F. Stone. Valuable assistance received through the Graduate School is also acknowledged. Miss R. M. Steelman and Dr. J. D. Kurbatov kindly carried out the chemical separations.

[6] Hibdon, Pool, and Kurbatov, Phys. Rev. 67, 289 (1945).

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Structure in the X-Ray *K* Absorption Edges of Solid Potassium Chloride

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A two crystal vacuum spectrometer was used to investigate the x-ray absorption spectrum of solid potassium chloride over a fifty-volt range in the vicinity of the *K* edges of both potassium and chlorine. Absorbers were prepared by vacuum evaporation of the salt onto thin, taut Kodapak films. High precision of intensity measurement was made possible by the use of fortuitously located $L\alpha_1$ radiations from targets of antimony and palladium. No corrections to observed intensities were made for the finite resolving power of the instrument. Results are expressed by graphs of relative absorption coefficient as a function of relative photon energy, and are summarized in a table of relative energy separations of prominent absorption maxima and minima.

INTRODUCTION

THE study of fine structure in the x-ray absorption edges of solids has long been considered important for an understanding of their electronic states. Recently, successful correlations between theory and experiment have been made for some of the simpler metals.[1,2] However, both theoretical and quantitative experimental work on non-conductors is meager and unsatisfactory. The present experimental work, using the most precise techniques known, was undertaken in order to provide information about the x-ray absorption fine structure of potassium chloride, an ionic crystal which is presumably one of the simplest from a theoretical standpoint.[3]

The experimental equipment consisted of a high voltage d.c. power supply, an x-ray tube, a set of potassium chloride absorbing films, a two crystal vacuum x-ray monochromator, an ionization chamber for detecting the x-rays, and a d.c. amplifier with galvanometer output for measuring ion currents. Several runs of absorber opacity *versus* monochromator setting were made for each absorption edge. The natural logarithms of the opacities for each run were then suitably corrected and the values for separate runs adjusted and matched so as to yield results conveniently represented as graphs of relative absorption coefficients as a function of relative photon energy.

[1] W. W. Beeman and H. Friedman, Phys. Rev. **56**, 392 (1939).
[2] J. B. Platt, Phys. Rev. To be published.
[3] Salient features of the fine structure of potassium chloride have been reported by J. M. Nuttall, Phys. Rev. **31**, 742 (1928); and G. P. Brewington, Phys. Rev. **46**, 861 (1934).

EXPERIMENTAL DETAILS

Apparatus

The high voltage power supply was of the full-wave rectifier type with filtered and electronically stabilized output.[4] The x-ray tube voltmeter, consisting of a galvanometer in series with two 1000-megohm resistors could be read with a precision of ± 0.1 percent and with an accuracy of ± 0.5 percent at 10,000 volts. The tube current was read with a 0–100 milliammeter with a precision of ± 0.2 percent. The majority of measurements were made with a tube voltage of 10,000 volts and current of 100 ma.

Details of the x-ray tube construction have been given elsewhere.[5] The target used for the study of the potassium edge was an alloy of 12 percent antimony and 88 percent lead tinned to the copper target carriage,[2] while the target used for the study of the chlorine edge was a 10-mil sheet of palladium soldered to the carriage. These target materials were selected because the $L\alpha_1$ lines of antimony and palladium provided both high x-ray intensity in the regions of the edges and standards for wave-length measurement.

The absorbing films were prepared by vacuum evaporation of chemically pure potassium chloride from a furnace, similar to one described by Schneider and O'Bryan,[6] onto tautly stretched Kodapak films 0.001 inch thick. The thickness of

[4] L. G. Parratt and J. W. Trischka, Rev. Sci. Inst. **13**, 17 (1942).
[5] L. G. Parratt, Phys. Rev. **54**, 99 (1938).
[6] E. G. Schneider and H. M. O'Bryan, Phys. Rev. **51**, 293 (1937).

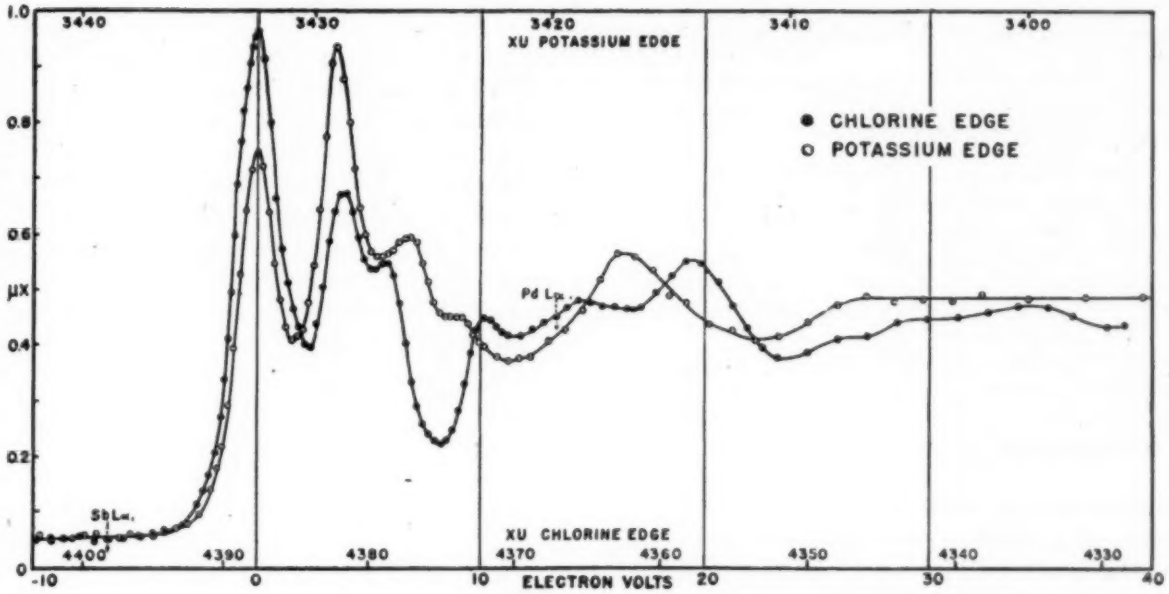

FIG. 1. X-ray K absorption edge of chlorine and potassium in solid potassium chloride.

the potassium chloride films so prepared was calculated to be approximately 35,000 Angstroms. Transmission of x-rays through each of several areas of each absorber showed that they were sufficiently uniform to make the exponential absorption law valid within 0.1 percent or better.

The two-crystal vacuum spectrometer has been described elsewhere.[5] Results of measurements of the (1, −1), (1, +1), (2, −2) widths and (1, −1) percent reflection for Cu $K\alpha$ radiation placed the crystals in Class I.[7] From the (1, −1) widths of the Sb $L\alpha_1$ and Pd $L\alpha_1$ radiations, the physical resolving power was estimated to be approximately 12,000, which was essentially the resolving power of the instrument, since the geometrical resolving power was of the order of 60,000.

To obtain 99 percent absorption of the x-ray beam the ionization chamber[2] was filled with krypton to a pressure of 20 cm of Hg for the study of the chlorine K edge and with argon to a pressure of 26 cm of Hg for the study of the potassium K edge. A modified Barth circuit,[8] utilizing a Western Electric D96475 tube, amplified the currents from the ionization chamber. Its sensitivity was 150,000 mm/volt on a scale two meters from the galvanometer.

[7] L. G. Parratt, Rev. Sci. Inst. **6**, 387 (1935).
[8] D. B. Pennick, Rev. Sci. Inst. **6**, 115 (1935).

X-Ray Intensities

In the region of the potassium edge the peak of the Sb $L\alpha_1$ line, at 10 kv and 100 ma, provided the greatest intensity, giving an ionization current of 1.2×10^{-12} ampere or a galvanometer deflection of 5620 cm on the highest sensitivity (grid resistor of 10^{12} ohms). The weakest direct beam used, continuous radiation, gave a current of 1.4×10^{-14} ampere or a 64-cm deflection at highest sensitivity. In the region of the chlorine edge the Pd $L\alpha_1$ line gave a peak intensity at 10 kv and 80 ma of 3.5×10^{-12} ampere, corresponding to a 16,700-cm galvanometer deflection at full sensitivity. The weakest direct beam gave a current of 2.5×10^{-14} ampere or a deflection of 112 cm at full sensitivity.

Procedure and Analysis of Data

Because of the rapidly varying intensity of the line radiation with crystal angle, data for a complete opacity measurement were taken at each wave-length setting of the monochromator according to the schedule outlined by Platt.[2] The time required for each opacity measurement varied from 10 to 14 minutes depending on the amplifier grid resistor used. Of the five absorbers prepared, two were used for recording both absorption edges. Eight runs with four different

TABLE I. Energy separations in electron volts of fine structure maxima and minima from initial absorption peak.

Feature	K−*K* edge	Cl−*K* edge
A	0.0	0.0
α	1.7	2.3
B	3.5	3.8
β	5.5	5.2
C	6.9	5.7
γ	?	8.2
D	9.0	10.1
δ	11.5	11.4
E	16.3	14.3
ε	22.4	?
F	27.5	19.2
ζ	—	23.4
G	—	29.9

absorbers were made in the region of the potassium edge. Of these runs three covered the entire region studied. In the region of the chlorine edge three runs, with three different absorbers, were made.

The general method of treating absorption measurements where more than one element is present in the absorber has been described by Platt.[8] The jump ratios of both potassium and chlorine were taken to be 10. Successive runs were matched at the peak of the first absorption line. The corrections for higher order radiation, made to the natural logarithm of the opacities, had a maximum value of 3 percent for the potassium measurements and 1 percent in the region of the chlorine edge. No correction for the resolving power was attempted since no reliable method has been developed for making such corrections to absorption spectra.

For purposes of absolute wave-length determination, separate runs were made to determine the position of the first absorption maximum relative to the $L\alpha_1$ radiation used for recording the fine structure.

RESULTS

Figure 1 is a graph of relative values of μx *vs.* relative photon energy in electron volts in the regions of the *K* absorption edges of chlorine and potassium in solid potassium chloride. The wave-length scale used in expressing the results is an X.U. scale rather than a Siegbahn scale. The standard wave-lengths used were 3431.8 Siegbahn units for Sb $L\alpha_1$ and 4358.5 Siegbahn units for Pd $L\alpha_1$.[9] The ratio of the grating and Siegbahn scales of wave-lengths was taken as 1.002034.[10] The wave-lengths of the peaks of the first absorption lines in the chlorine and potassium spectra were determined to be 4387.6 X.U. and 3432.5 X.U. respectively.[11] In establishing the voltage scale the wave-length associated with one absolute volt was taken to be $12{,}395\times10^{-8}$ cm absolute volt.[10]

For ease of comparison with other data and with theoretical results, which are likely to take the form of energy separations, Table I presents the energy separations from the initial maximum of the various maxima and minima in Fig. 1. Maxima and minima are referred to by letters of the Roman and Greek alphabets respectively.

It is estimated from the self and mutual consistencies of the various runs that variations in μx as small as 3 percent are significant. An addition of component errors gives a maximum uncertainty in absolute wave-length specification of ±0.15 X.U. Relative wave-lengths are precise to ±0.02 X.U., the error in matching different runs.

The author wishes to express his indebtedness to Professor L. G. Parratt for many stimulating discussions and helpful criticisms, and to Professor H. A. Bethe for suggesting the investigation.

[9] H. Hjalmar, Zeits. f. Physik **3**, 262 (1920).
[10] R. T. Birge, Rev. Mod. Phys. **13**, 233 (1941).
[11] Or 4378.7 Siegbahn units and 3425.5 Siegbahn units respectively.

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Theoretical Continuous X-Ray Energy and Polarization

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By the theory of Sommerfeld, relativity effects and retardation of potential being neglected, matrix elements and associated components of continuous spectrum x-radiation are computed for a variety of electron-nucleus collisions and for a distributed series of positions in the spectrum. The calculations cover values of V/Z^2 from 0.06128 to 3.356, where Z is the atomic number of the nucleus, and V is the bombardment potential in electrostatic units. Accuracy of calculation is 1 percent. Screening is neglected except at the long wave limit of the spectra where it is taken into account by Sauter's method. Empirical algebraic formulas are found which closely represent the rigorously calculated results. Intensity and polarization predictions for any direction of emission and any excitation conditions within the range of applicability of the theory may be readily drawn from the computed results. Elwert's proposed correction factor for rectifying the approximate spectral intensities of Sommerfeld and Maue is found effective within the limits of its restricting assumptions. Theoretical efficiencies of continuous x-ray production are calculated by combining theoretical intensities with known rates of electron energy loss in traversing matter. Thick target efficiency (a ratio, not percent) is given by $1.4\times10^{-6}Z$ kv. Thin target efficiency is found to be approximately twice the thick target efficiency for any given Z and kv.

THE problem of the continuous x-ray spectrum is to find answers to such questions as the following: When an electron of stated kinetic energy collides with a thin layer of matter of atomic number Z and of some given surface density, what is the probability that a photon whose frequency lies between any stipulated limits shall be emitted in an elementary solid angle of specified magnitude oriented in any defined manner with respect to the direction of motion of the bombarding electron? A further question concerns the state of polarization of such a photon.

Answers to these questions are implicit in extant theory, but in only a very few cases have explicit numerical answers been extracted from the theories and placed on the record. It is not yet possible to say broadly that the theory either does or does not agree with experimental facts, though a few encouraging similarities have appeared. We shall not have a satisfying number of such comparisons without further work on both sides of the gap, i.e., until many more good measurements have been completed and until theoretical predictions are made available *in numerical form*.

The present paper attempts to supply the second of these necessities. It contributes nothing to the basic theoretical picture according to which the emission of the continuous x-ray spectrum is now understood; rather, it presents a computational procedure for obtaining numerical values of x-ray energy and polarization from the theory, together with the results of such calculations.

For a system consisting of a bare nucleus and one electron, Schrödinger's equation predicts a continuous range of *positive* energy states, and a continuous spectrum must result from the joint operation of the possible transitions between such states. The spectral features may be deduced, as Oppenheimer[1] and Sugiura[2] have shown, by calculating matrix elements from characteristic functions corresponding to states involved in the transitions, but the method introduced by Sommerfeld[3] and developed by a number of subsequent theorists has greater analytical simplicity and will be followed in the present paper.

The theory of Sommerfeld considers the wave systems of an electron approaching an atomic nucleus and departing with altered direction and reduced speed, having suffered an energy loss which appears as the energy of the emitted photon. The single process can be described with rigor, but the integration over all possible direc-

[1] J. R. Oppenheimer, Zeits. f. Physik **55**, 725 (1929).
[2] Y. Sugiura, Sci. Pap. Inst. Phys. Chem. Res. Tokyo **17**, 89 (1931); and earlier contributions referred to therein.
[3] A. Sommerfeld, Ann. d. Physik [5] **11**, 257 (1931); O. Scherzer, Ann. d. Physik [5] **13**, 137 (1932); F. Sauter, Ann. d. Physik [5] **18**, 486 (1933), [5] **20**, 404 (1934); A. Sommerfeld and A. W. Maue, Ann. d. Physik [5] **23**, 589 (1935); G. Elwert, Ann. d. Physik [5] **34**, 178 (1939); R. Weinstock, Phys. Rev. **61**, 585 (1942), **65**, 1 (1944).

tions of departure of the electron—a necessary operation if theory is to be compared with experiment—has been found troublesome by all who have tried it.

Approximations, physical or mathematical, have been necessary features of all the treatments. The present results are conditioned by the inexact physical assumptions of the original Sommerfeld[3] theory, but we follow Weinstock[3] in avoiding mathematical approximations such as those which restricted the applicability of the results of Sauter and of Elwert.[4]

The questionable physical assumptions and approximations are the following: (a) Except in connection with the long wave limit of the spectrum, atomic electrons are ignored, and the nucleus is assumed to be the center of a pure Coulomb field. (b) Relativity effects, retardation of potential, and the effects of electron spin are neglected. (c) The de Broglie waves of the approaching and receding electron are treated as plane waves, filling all space.

Picture a row of electrons moving along or closely parallel to the x-axis of a cartesian coordinate system (see Fig. 8), approaching the origin from the negative side with speed v_1 conferred by a potential difference V. Upon arriving at the origin, the electrons strike or pass through a plane target coinciding with the yz-plane which is composed of atoms of atomic number Z in a uniform distribution. We investigate the continuous x-radiation resulting from collisions at the origin and departing in a direction lying in the xz-plane and making an angle θ with the positive x-direction. The electron leaves the collision with speed v_2. As applied to such a picture, the theory deals in probabilities of photon emission, but with large numbers of atoms or of electrons, such probabilities increase proportionately, and one may speak with statistical accuracy of emitted intensities, powers, and energies.

The bombarded and emitting atoms simulate a source possessing calculable component Hertzian dipole moments, each of which produces radiation components I_x, I_y, and I_z, proportional to the square of the corresponding moment. The total intensity proceeding in a direction θ consists in general of contributions from all three radiation components weighted with reference to the angles between their dipole axes and the direction of observation. The observable intensity is given by[5]

$$I_\theta = I_x \sin^2\theta + I_y + I_z \cos^2\theta. \tag{1}$$

The calculation and discussion of $I_{x,y,z}$ are the objects of this paper; these components will be defined in such a way that I_θ (not rigorously a radiation intensity) comes out in the units ergs per steradian per unit frequency range per bombarding electron per atom-per-square-centimeter of target area. Neglect of electron spin leaves the direction of bombardment an axis of complete symmetry, so $I_y = I_z$ necessarily. There remain two quantities, I_x and I_y, to calculate, and each is a function of the four parameters θ, V, Z, and the frequency ν, a group which will be found reducible to three. In terms of these components, the intensity and polarization of radiation emitted in the direction θ are given by

$$\left.\begin{aligned} I_\theta &= I_x \sin^2\theta + I_y(1+\cos^2\theta) \\ P_\theta &= \frac{1-(I_y/I_x)}{1+(I_y/I_x)(2\csc^2\theta - 1)} \end{aligned}\right\}. \tag{2}$$

The method of Sommerfeld yields:

$$I_{x,y,z} = \frac{4\pi e^6 Z^2 (Ve - h\nu) N^2_{x,y,z}}{[1-\exp(-4\pi^2 e^2 Z/v_2 h)][\exp(4\pi^2 e^2 Z/v_1 h) - 1][(Ve)^{\frac{1}{2}} - (Ve - h\nu)^{\frac{1}{2}}]^4 mc^3}, \tag{3}$$

where $N^2_{x,y,z}$ are dimensionless quantities equivalent to $M^2_{x,y,z}/|A|^2$ of Weinstock's first reference. The computation of these quantities is almost the whole of the problem confronting the computer.

[4] Sauter is limited by Born's first approximation, $Ze^2/\hbar v_{1,2} \ll 1$; while Elwert's method is restricted by the assumption $(Ze^2/\hbar v_2) - (Ze^2/\hbar v_1) \ll 1$ or $Ze^2/\hbar v_1 \ll 1$. The effects of these conditions will be noted on a subsequent page.

CALCULATION OF N_y^2 $(=N_z^2)$

From Weinstock's derivation one finds

$$N_y^2 = \frac{16\pi^3 Z^2 e^3 m}{h^2 V \xi_0^3}\left[1 + \frac{2\pi^2 Z^2 e^4 m}{h^2(Ve - h\nu)}\right] \times \int_0^{w_0}\left\{\frac{\xi_0 w}{1-w} + \frac{w^2}{(1-w)^2}\right\}|G_3|^2 dw. \tag{4}$$

[5] G. Elwert, reference 3; p. 194.

New symbols appearing in Eq. (4) are defined as follows:

$$\xi_0=-4n_1n_2/(n_1-n_2)^2, \quad w_0=\xi_0/(\xi_0-1).$$

The quantity G_3 is the hypergeometric function $F(1+n_1, -n_2, 2, w)$ where n_1 and n_2 are imaginary quantities defined by $n_{1,2}=(-2\pi e^2Zi)/(hv_{1,2})$. The physical significance of w is of no concern since it goes out in the definite integration.

Weinstock expanded G_3 in series and integrated Eq. (4) term by term, obtaining his compact Eq. (9). Although this works well in some cases, it has not been found generally trustworthy because of the danger that the series of integrals may be chopped off too soon. It is quite impossible to depend upon appearances in estimating the progress of its convergence, and no positive criterion has been found. It is known that as many as twenty terms are insufficient in certain cases to reduce the residue to negligibility, so the evaluation is both suspect and laborious. It was decided, therefore, to evaluate the integrand in Eq. (4) for representative values of w and to carry out the integration graphically.

The range of integration extends from zero to w_0, a positive quantity lying between zero and unity. For those cases (values of V and ν) which give values of w_0 not too near to unity, say not exceeding 0.75, the necessary values of G_3 may be obtained from the standard hypergeometric expansion

$$F(a, b, c, x)\equiv 1+\frac{abx}{1c}+\frac{a(a+1)b(b+1)x^2}{1\cdot 2\cdot c(c+1)}$$
$$+\frac{a(a+1)(a+2)b(b+1)(b+2)x^3}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}+\cdots\equiv\sum_{\mu=0}^{\mu=\infty}g_\mu x^\mu$$

where μ is an integer designating a series term, which starts with $\mu=0$ for the first term. The coefficients are given by

$$g_0=1, \text{ and in general}$$
$$g_{\mu+1}=\frac{(\mu+a)(\mu+b)}{(\mu+1)(\mu+c)}g_\mu.$$

By use of this series, G_3 may usually be evaluated with an uncertainty not greater than one percent without going beyond 10 or 15 terms. For values of w too large for prompt convergence, use is made of the identity[6]

$$G_3=C_1F(1+n_1, -n_2, n_1-n_2, 1-w)$$
$$+C_2F(1-n_1, 2+n_2, 2-n_1+n_2, 1-w),$$

where

$$C_1=\frac{-n_1+n_2}{-n_1n_2(1+n_2)}\bar{\Delta},$$

$$C_2=\frac{\Delta}{n_1(-1+n_1-n_2)}(1-w)^{1-n_1+n_2},$$

$$\Delta=\Gamma(n_1-n_2)/\Gamma(n_1)\Gamma(-n_2),$$

and $\bar{\Delta}$ designates the complex conjugate of Δ.

Numerical evaluation of Δ is most readily carried out by means of the identity

$$1/\Gamma(x)=x(x+1)$$
$$\times(1+B_1x+B_2x^2+B_3x^3+\cdots). \quad (5)$$

The coefficients of x as calculated by Bourguet[7] are reproduced in Table I.

The terms in the series of Eq. (5) are alternately imaginary and real, leading to a complex Δ. For the largest values of n_2, the tabulated coefficients are insufficient, and it is advantageous to use Legendre's duplication formula

$$\pi^{\frac{1}{2}}\Gamma(2x)=2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2}),$$

to bring the arguments of the gamma-functions within the range of the table. The imaginary character of n_1 and n_2 requires special attention in the evaluation of C_2; here we may write

$$(1-w)^{1-n_1+n_2}=(1-w)(\cos\theta+i\sin\theta),$$

in which

$$\theta=|n_1-n_2|\log(1/1-w).$$

If the integrand of Eq. (4) be plotted as a function of w, the curve will usually be found inconvenient for planimeter integration because of the high and narrow peak at the upper limit. This feature is avoided by plotting as a function

[6] E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable* (Oxford University Press, New York, 1935), p. 251.

[7] L. Bourguet, Acta Math. 2, 261 (1883). Harold T. Davis, *Tables of the Higher Mathematical Functions* (The Principia Press, Inc., Bloomington, Indiana, 1933), Vol. 1, p. 185.

of u, where $u=1/(1-w)$, in which case the errors of area measurement may well be less than those of computation. The transformed integral is

$$\int_1^{1-\xi_0} \frac{(u-1)(\xi_0+u-1)}{u^2}|G_3|^2 du.$$

As ξ_0 increases, the maximum of the integrand becomes higher and narrower, but plotting in two or three different sections with suitable changes of scale readily keeps it in hand.

CALCULATION OF N_z^2

The calculation of N_z^2 might be carried out in a manner analogous to that described above, but Weinstock's expression contains three definite integrals, and although he has shown how to replace them by infinite series, the problem of convergence persuades us to adopt a differing approach. Sommerfeld and Maue have shown how to express the sum $N^2=N_x^2+N_y^2+N_z^2$ in closed form, and Weinstock has put their expression into a form based upon four rapidly converging series, two of which we find to be identical. We therefore employ Weinstock's equation for N^2 in the form shown below and later solve for $N_x^2=N^2-2N_y^2$. This avoidance of a direct attack upon N_x^2 follows, in principle, the example of Elwert.

TABLE I. Coefficients in the power-series expansion

$$\frac{1}{\Gamma(x)}=x(x+1)(1+B_1x+B_2x^2+\cdots).$$

n	B_n			
1	−0.4227	8433	5098	4671
2	−0.2330	9373	6421	7867
3	0.1910	9110	1387	6915
4	−0.0245	5249	0005	4000
5	−0.0176	4524	4550	1443
6	0.0080	2327	3022	2673
7	−0.0008	0432	9775	6044
8	−0.0003	6083	7816	2548
9	0.0001	4559	6142	1399
10	−0.0000	1754	5859	7517
11	−0.0000	0258	8995	0224
12	0.0000	0133	8501	5466
13	−0.0000	0020	5474	3152
14	−0.0000	0000	0159	5268
15	0.0000	0000	6275	6218
16	−0.0000	0000	1273	6143
17	0.0000	0000	0092	3397
18	0.0000	0000	0012	0028
19	−0.0000	0000	0004	2202
20	0.0000	0000	0000	5240
21	−0.0000	0000	0000	0140
22	−0.0000	0000	0000	0067

The working equation for N^2 is

$$N^2=\frac{-16\pi}{\xi_0}\left[\left|\frac{\Delta}{n_1}\right|^2\sum_{r=1}^{\infty} rq_r\xi_0^{-r-1} - \Re\left\{\frac{\Delta^2}{n_1n_2}(-\xi_0)^{n_1-n_2}\left[\sum_{r=1}^{\infty} rj_r\xi_0^{-r-1} - (n_1-n_2)\sum_{r=0}^{\infty} j_r\xi_0^{-r-1}\right]\right\}\right], \quad (6)$$

wherein $\Re\equiv$"real part of," and the quantities q_r and j_r are themselves defined by the finite series:

$$q_r=\sum_{\nu=0}^{r} g_{4,\nu}g^*_{4,r-\nu}; \quad j_r=\sum_{\nu=0}^{r} g_{4,\nu}g^*_{5,r-\nu}.$$

Quantities in these series are defined and calculated by the recursion relations:

$$g_{4,0}=g_{5,0}=1,$$

$$g_{4,\nu}=\frac{(\nu-1-n_1)^2}{\nu(\nu+n_2-n_1)}g_{4,\nu-1},$$

$$g_{5,\nu}=\frac{(\nu-1-n_2)^2}{\nu(\nu+n_1-n_2)}g_{5,\nu-1}.$$

The asterisk (*) denotes the conjugate function. In these series ν and r are real integers assuming consecutive values within the stated ranges. The convergence is such that as few as three terms sometimes suffice to fix the value of a series. The g's are, of course, complex, but the conjugates in q_r and j_r cancel out the imaginaries and leave these quantities and their series real.

NUMERICAL EVALUATION

Inspection of the foregoing equations shows that the variables V, Z, and ν are present only in combinations reducible to V/Z^2 and ν/ν_0. This reduction of the number of parameters controlling the spectral features from four to three (the two quotients above and θ) greatly reduces the amount of computation necessary to cover the full range of experimental variables, since any one complete calculation furnishes a separate prediction concerning each of the 92 elements.

TABLE II. Components and total energy of radiation.

V/Z^2	ν/ν_0	$I_x\times10^{30}$	$I_y(=I_z)\times10^{30}$	$W\times10^{30}$	V/Z^2	ν/ν_0	$I_x\times10^{30}$	$I_y(=I_z)\times10^{30}$	$W\times10^{30}$
3.356	0.403	0.0809	0.0296	1.17	0.06041	1.00	5.51	0.718	58.2
						0.624	5.12	1.35	65.4
2.171	0.623	0.128	0.0228	1.45		0.443	4.86	1.79	70.6
						0.217	4.40	2.81	83.9
1.776	1.00	0.146	0.00132	1.24		0.00	1.72	5.81	112.0
	0.761	0.154	0.016	1.56					
	0.197	0.144	0.105	2.97	0.03776	1.00	8.48	1.30	92.9
	0.0962	0.136	0.151	3.67		0.708	8.11	1.92	100.0
	0.00	0.0586	0.223	4.23		0.346	7.34	3.32	117.0
						0.00	2.78	9.10	176.0
1.351	1.00	0.207	0.00223	1.77					
					0.03546	1.00	9.00	1.40	98.9
0.7896	1.00	0.398	0.00761	3.47		0.761	8.73	1.92	105.0
	0.624	0.377	0.072	4.37		0.197	7.35	4.76	141.0
	0.443	0.361	0.121	5.05		0.0962	6.75	6.34	163.0
	0.217	0.332	0.227	6.59		0.00	2.95	9.60	186.0
	0.00	0.132	0.496	9.42					
					0.02723	1.00	11.5	1.92	128.0
0.4934	1.00	0.685	0.0191	6.06		0.624	10.8	3.01	141.0
	0.708	0.637	0.0948	6.92		0.443	10.4	3.84	152.0
	0.346	0.578	0.254	9.10		0.217	9.74	5.68	177.0
	0.00	0.211	0.776	14.8		0.00	3.81	12.2	236.0
0.3493	1.00	0.999	0.0378	9.01	0.02265	1.00	13.62	2.37	154.0
						0.577	12.73	3.81	171.0
0.2961	1.00	1.20	0.0518	10.9		0.00	4.60	14.5	282.0
	0.577	1.05	0.250	13.0					
	0.00	0.351	1.26	24.1	0.01702	1.00	17.7	3.27	204.0
						0.708	17.2	4.31	216.0
0.1709	1.00	2.12	0.141	20.1		0.346	16.0	6.84	248.0
						0.00	6.09	18.8	366.0
0.1359	1.00	2.63	0.214	25.6					
	0.761	2.43	0.402	27.1	0.01576	1.00	19.0	3.58	219.0
	0.197	1.91	1.39	39.3		0.624	18.2	5.14	238.0
	0.0962	1.79	1.90	46.8		0.443	17.7	6.36	255.0
	0.00	0.765	2.705	51.7		0.217	16.6	9.25	294.0
						0.00	6.58	20.2	394.0
0.06128	1.00	5.43	0.697	57.2					
	0.761	5.18	1.05	61.0	0.009854	0.708	28.9	7.48	368.0
	0.197	4.26	2.92	84.5		0.346	27.2	11.2	415.0
	0.0962	3.92	3.95	99.0		0.00	10.5	30.5	599.0
	0.00	1.70	5.74	111.0					

Many of these predictions, however, relate to definitely relativistic voltage or frequency ranges and are, therefore, not fully applicable.

Values of I_x and $I_y=I_z$, calculated by the methods described above are listed in Table II, along with the related values of V/Z^2 and ν/ν_0. Birge's[8] 1941 values of the atomic constants have been used. Since the I's vary approximately as the ninth power of e, precision here is not as superfluous as might be assumed. Indeed, an error in e as great as the discrepancy between today's value and that of a decade ago could easily throw the calculated intensities off by much more than the errors of intensity measurement.

[8] R. T. Birge, Rev. Mod. Phys. **13**, 233 (1941).

The quantity W, introduced in the fifth and last columns of Table II, is the total energy of continuous x-radiation in all directions. Integration of I_θ of Eq. (2) over the 4π steradians of space surrounding the point of collision gives at once $W=(8\pi/3)[I_x+I_y+I_z]$, a function of ν given in the units ergs per unit frequency interval per bombarding electron per atom-per-square-centimeter of target area.

The numerical values in Table II have been checked against each other by many graphical tests, and indeed, such checks were used at all stages of the calculations as a guard against errors of computation. Although each row of the table (except at $\nu/\nu_0=0$ and some at $\nu/\nu_0=1$) represents over 1200 elementary computing operations, it is believed that computing errors

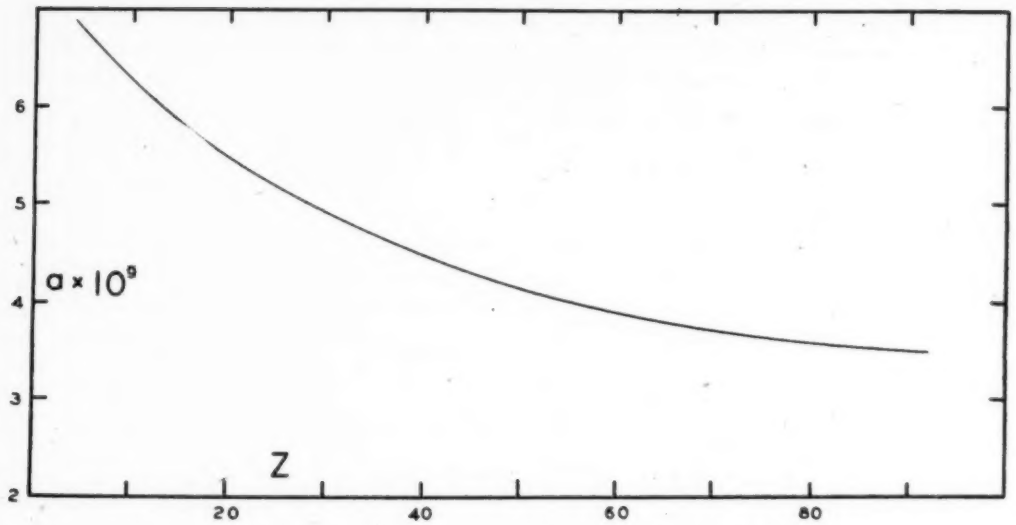

FIG. 1. Screening constants a (in cm) of the Sauter atomic potential distribution $V_s=(-eZ/r)\exp(-r/a)$. Values shown by curve give best agreement with Fermi atomic potentials.

and approximations have in no case impaired a calculated result by more than one percent.

Low Frequency Limit

The computations described above have been subject to the assumption that the bombarded nucleus is effectively unscreened. This does not preclude the application of the results to real nuclei,[9] fully clothed with screening electrons, as long as the acceleration of the bombarding electrons takes place inside of the surrounding electron screen. This is probably what happens in the production of most of the spectrum, but it definitely does not always happen when radiation of vanishingly small quantum energy is emitted. Such radiation requires the application of such slight forces to the bombardment electron that it may be produced by an electron passing a bare nucleus at a great distance. The collision cross section for radiation of zero frequency is very great, and one readily sees why the Sommerfeld theory predicts infinite intensity at $\lambda=\infty$. At this end of the spectrum, it is necessary to assume some screening if predictions applicable to real atoms are desired.

We have, therefore, calculated $I_{x,y}$ at $\nu=0$ by the method of Sauter, who shows how to compute the radiation from an atom whose potential distribution is given by $V_S=(-eZ/r)\exp(-r/a)$, where r is a radial coordinate originating at the atomic nucleus. In the unscreened case ($a=\infty$), Sauter's method gives intensity values which are much too low in the high frequency region of the spectrum, but Elwert has shown that Sauter's method agrees with the Sommerfeld method in

TABLE III. Radiation components at $\nu=0$ for screened nuclei.

V	Z	$I_x\times10^{30}$	$I_y\times10^{30}$	V	Z	$I_x\times10^{30}$	$I_y\times10^{30}$
28.43	4	.0584	.190	33.99	30	2.77	8.14
44.40	5	.0585	.199	48.94	36	2.78	8.45
87.03	7	.0586	.217	83.40	47	2.78	8.93
143.9	9	.0586	.230	131.4	59	2.78	9.34
214.9	11	.0586	.240	201.2	73	2.79	9.78
300.2	13	.0586	.248	299.1	89	2.79	10.2
28.42	6	.131	.418	300.2	92	2.93	10.8
50.54	8	.132	.451				
78.96	10	.132	.476	33.36	35	3.80	11.1
133.4	13	.132	.504	50.34	43	3.81	11.6
202.1	16	.132	.527	82.36	55	3.81	12.2
315.8	20	.132	.544	133.4	70	3.81	12.8
				201.4	86	3.82	13.4
31.58	8	.210	.673	230.5	92	3.82	13.6
49.34	10	.211	.713				
83.40	13	.211	.758	50.04	47	4.57	2.99
142.6	17	.211	.803				
197.3	20	.211	.828	32.95	44	6.07	17.2
308.4	25	.211	.864	49.64	54	6.08	18.0
				83.40	70	6.09	19.1
50.04	13	.351	1.17	144.0	92	6.10	20.3
34.76	16	.763	2.39	33.36	46	6.56	18.6
49.07	19	.764	2.47	49.44	56	6.57	19.4
84.93	25	.765	2.65	84.00	73	6.58	20.6
139.1	32	.765	2.77	133.4	92	6.59	21.8
206.7	39	.765	2.87				
300.2	47	.766	2.97	33.15	58	10.5	28.7
				49.67	71	10.5	30.3
300.2	70	1.70	6.36	83.40	92	10.5	32.3
34.79	24	1.72	5.21				
50.80	29	1.72	5.46				
82.69	37	1.72	5.73				
133.4	47	1.72	5.98				
203.2	58	1.72	6.21				
296.0	70	1.72	6.44				

[9] A. L. Hughes, Phys. Rev. **55**, 350 (1939), shows that atomic electrons play a negligible part in the *elastic* scattering of fast electrons by atoms.

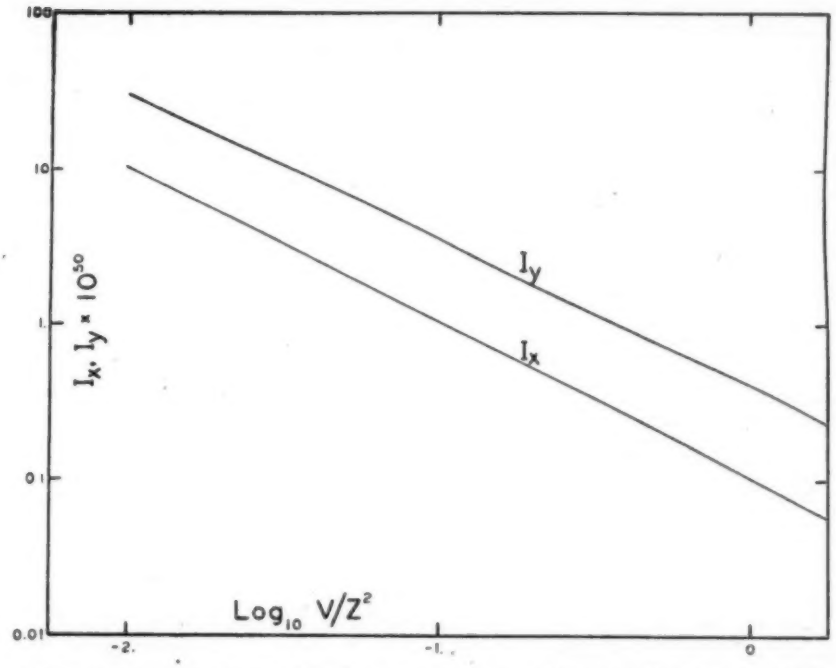

FIG. 2. Radiation components from Table II at the long wave limit ($\nu=0$). (Note both axes are logarithmic.) $I_y > I_x$ means negative polarization.

its approach to infinite intensities at the low frequency limit. It seems reasonable therefore, to seek a proper value of a and apply the method at the long wave limit.

The values of a adopted were those which brought the Sauter potential distribution into best agreement with the Fermi[10] atomic potential $V_F = Ze\varphi/r$ over the region of r for which the Fermi expression is defined. Figure 1 shows the variation of a with Z.

Equation (20) of Sauter's[11] first paper may be specialized for $\nu=0$ and decomposed into

$$\left.\begin{aligned} I_x &= \frac{e^5}{c^3 m(V/Z^2)}\left[\frac{-2h^2}{8mVea^2}\log\frac{8mVea^2}{h^2}+1\right] \\ I_y &= \frac{e^5}{c^3 m(V/Z^2)} \\ &\quad\times\left[\left(\frac{1}{2}+\frac{h^2}{8mVea^2}\right)\log\frac{8mVea^2}{h^2}-1\right] \end{aligned}\right\}. \quad (7)$$

These values of I_x and I_y are not functions of V/Z^2 alone but depend also upon V. This dependence is very slight for I_x, a variation of V by a factor of 10 (V/Z^2 remaining constant) affecting I_x by much less than one percent. In the case of I_y, the corresponding variation goes as high as 25 percent. The difference is classically understandable since the force between a nucleus and a remote passing electron is principally in the component lying along the impact parameter. In plotting spectral features as functions of V/Z^2 and ν/ν_0, we have in some cases to follow used mean values of the low frequency limit I_x and I_y within the narrow scatter described above. Such mean values are entered in Table II, and Table III contains a fuller display of all the separate values calculated from Eqs. (7). These easily calculated values (Fig. 2) are in every case readily acceptable extensions of our hard-won curves for finite frequencies.

High Frequency Limit

No short cut is available for computing the short wave limit radiation components. The elaborate methods described above are applicable, but several of the series must be carried out to tedious lengths. The second reference of

[10] E. Fermi, Zeits. f. Physik **48**, 73 (1928). The quantities ϵ and φ are defined and tabulated in this paper.

[11] There are two mistakes in the equation as printed: R^0 should be R^2, and p_0 in the denominator outside the braces should be 3.

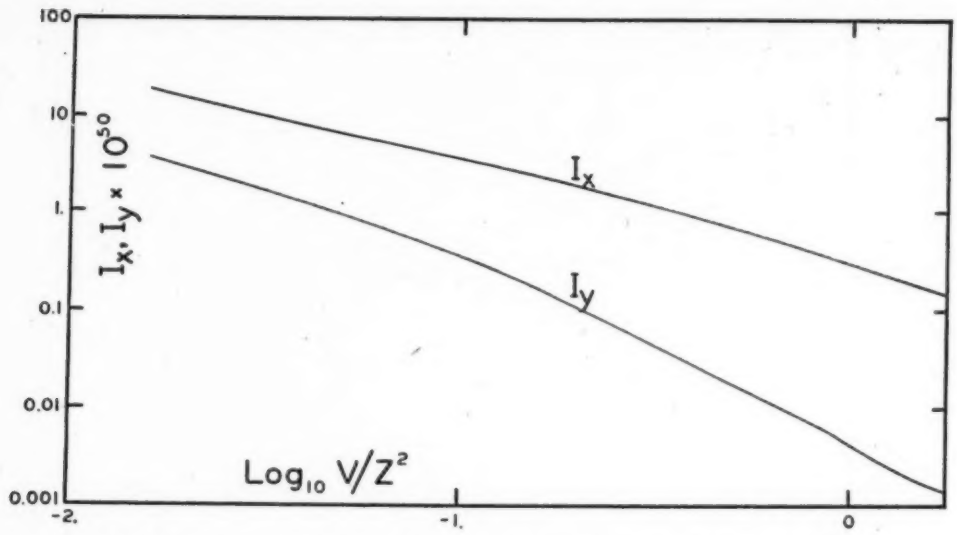

FIG. 3. Radiation components from Table II at the short wave limit ($\nu=\nu_0$). (Note both axes are logarithmic.)

Weinstock indicates the special forms which the working equations take at this limit, but no overall simplification results. We have, therefore, carried out the computations for only four cases at the limit. These have been supplemented by limit data obtained by extrapolation of results of related cases for which $\nu/\nu_0<1$. Since the radiation components are functions of two variables (V/Z^2 and ν/ν_0), it was possible to arrive at the same result by two quite independent extrapolations. If two such extrapolations yielded results in agreement to within one percent, the result was accepted as correct. Figure 3 shows the variation of I_x and I_y at the high frequency limit. These curves are important in their relation to the mooted question of high frequency limit polarization.

Approximate Representation

Detailed inspection of Table II shows that the values of I_x and I_y are, to a rough approximation, inversely proportional to V/Z^2. For I_x the approximation is close enough so that all the values within the range of our investigation may be stated with a mean error of 10 percent and a maximum error of 20 percent by the equation

$$I_x=3.0\times10^{-51}Z^2/V.$$

Near the short wave limit ($\nu/\nu_0=1$), the variation of I_x with V/Z^2 is not as simple, and the maximum errors are found. Far from the short wave limit, approximate representations become quite satisfactory; all values of I_x investigated at $\nu/\nu_0=0.2$ are given by $I_x=2.6\times10^{-51}Z^2/V$ with errors all under 2 percent. It should be realized that the word "error" as used here means only the discrepancy between the approximate expression and the more rigorously calculated values of Table II; it has no relation either to the errors of the Sommerfeld theory or the errors of computation.

The values of I_y and I_z are less easily summarized. They increase rapidly with increasing wave-length in a given spectrum whereas the general (but not invariable) tendency of I_x is to decrease slowly. At $\nu/\nu_0=0.4$ all values of $I_y(=I_z)$ are given with errors under 6 percent by the expression $1.1\times10^{-51}Z^2/V$, but elsewhere in the spectra, expressions of this type do not suffice.

In view of the enormous amount of work involved in computing spectral characteristics from the Sommerfeld equations or any of their modifications, it seems worth while to consider a second approximate representation of their results. Using primes to distinguish these approximations, we find:

$$I_x'\times10^{50}V/Z^2=0.252+a(\nu/\nu_0-0.135)-b(\nu/\nu_0-0.135)^2, \quad (8)$$

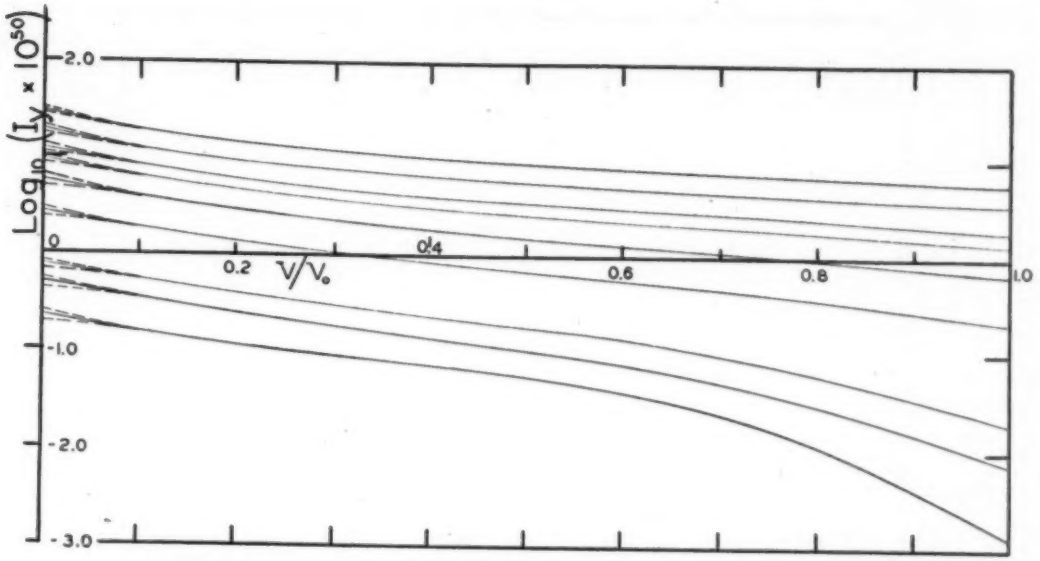

FIG. 4. Variation of I_y (or I_z) with position in the spectrum. Each curve pertains to a specific value of V/Z^2. Reading downward from the top curve, these values are 0.00985, 0.01576, 0.0272, 0.0355, 0.0613, 0.1359, 0.493, 0.790, and 1.776 electrostatic units of potential difference.

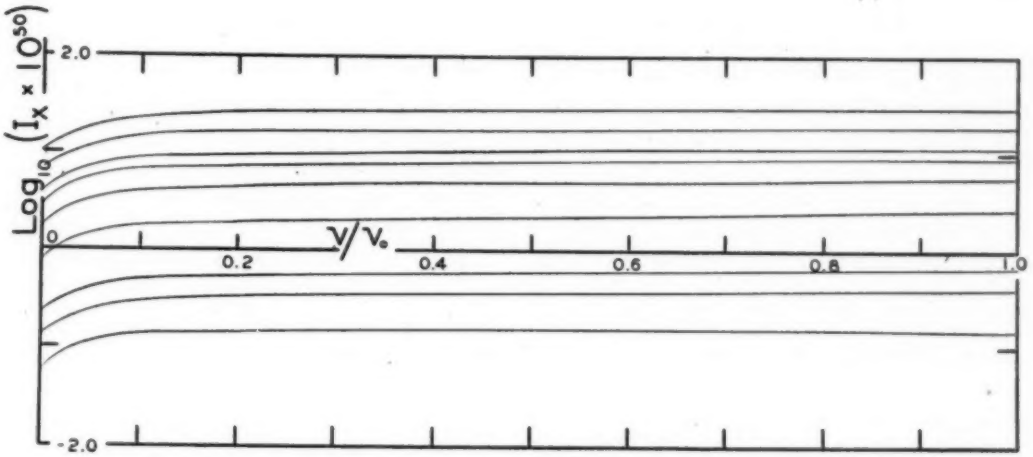

FIG. 5. Variation of I_x with position in the spectrum. Each curve pertains to a specific value of V/Z^2. Reading downward from the top curve these values are 0.00985, 0.01576, 0.0272, 0.0355, 0.0613, 0.1359, 0.493, 0.790, and 1.776 electrostatic units of potential difference.

where

$$a=1.47B-0.507A-0.833,$$

$$b=1.70B-1.09A-0.627,$$

$$A=\exp(-0.223V/Z^2)-\exp(-57V/Z^2),$$

$$B=\exp(-0.0828V/Z^2)-\exp(-84.9V/Z^2).$$

$$I'_{y,z}\times 10^{50}V/Z^2=-j+\frac{k}{(\nu/\nu_0)+h}, \qquad (9)$$

where

$$h=\frac{-0.214y_1+1.21y_2-y_3}{1.43y_1-2.43y_2+y_3},$$

$$j=(1+2h)y_2-2(1+h)y_3,$$

$$k=(1+h)(y_3+j),$$

$$y_1=0.220(1-0.390\exp(-26.9V/Z^2)),$$

$$y_2=0.067+0.023/((V/Z^2)+(0.75)),$$

$$y_3=-0.00259+0.00776/((V/Z^2)+0.116).$$

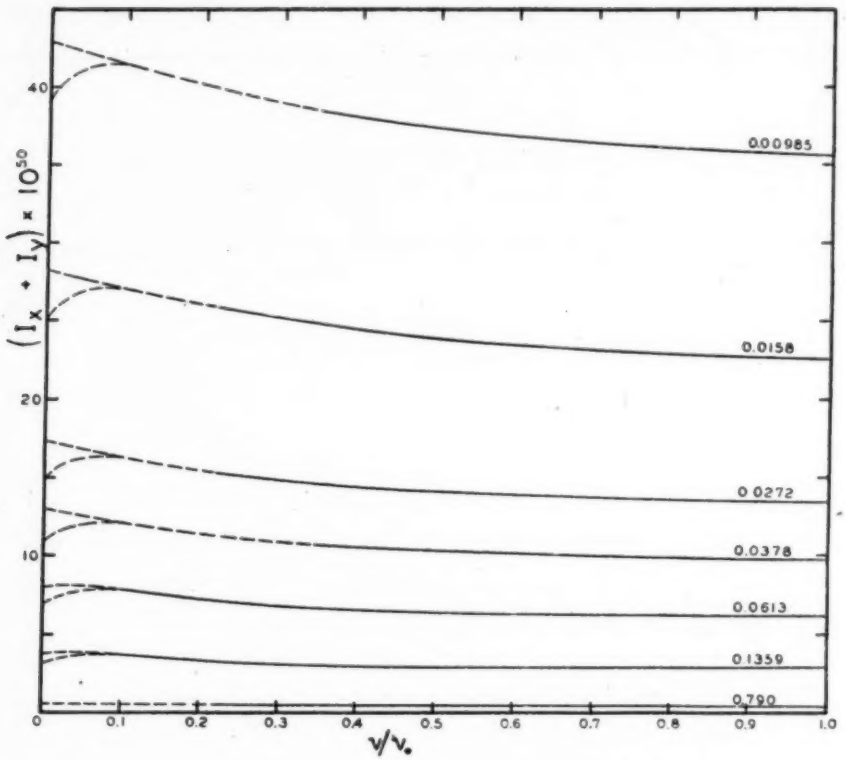

FIG. 6. Total intensity (ergs per steradian per unit frequency interval per bombarding electron per atom-per-cm²) emitted at 90° to the direction of bombardment as dependent upon position in the spectrum. Each curve pertains to the value of V/Z^2 shown just above it.

By these elementary expressions it is possible to compute the characteristics of continuous x-ray spectra with an expenditure of time and labor of the order of one percent of that required by the basic theory. Within the range covered by Table II, I_x', I_y', and I_z' differ from I_x, I_y, and I_z, respectively, by less than five percent at the worst and by about two percent on the average. The forms of Eqs. (8) and (9) have, naturally, no theoretical significance; the equations are merely empirical fits of the data of Table II.

Discussion of Table II

Figure 4, a plot of data from Table II, shows the variation of the y (or z) component of radiation with its controlling variables. In any given spectrum the component I_y (or I_z) diminishes in magnitude as the short wave limit ($\nu/\nu_0=1$) is approached. The diminution is most notable when targets of low atomic number are subject to bombardment by fast electrons. At the other end of the spectra, the Sommerfeld equations would have turned all the curves up toward infinite values. Introduction of screening in all calculations for which $\nu/\nu_0=0$ keeps the components finite but gives multiple values, since in this approximation the roles of V and Z do not always bring them together in the quotient V/Z^2. This accounts for the frayed appearance of the left-hand ends of curves of Fig. 4. It is presumed that if it were practicable to take correct account of screening throughout the spectrum, the curves of Fig. 4 would, throughout their lengths, consist of multiple bundles converging closely with increasing ν.

Figure 5 is a similar representation of I_x. The insensitivity of I_x to frequency (emphasized by logarithmic plotting) is the most conspicuous feature here. All points at $\nu=0$ have been calculated by Eq. (7), but screening has little effect on the long wave limit values of I_x and the splitting of the curves is too small to be shown.

The total radiation emitted in the direction $\theta=90°$ is given in accordance with Eq. (1) by

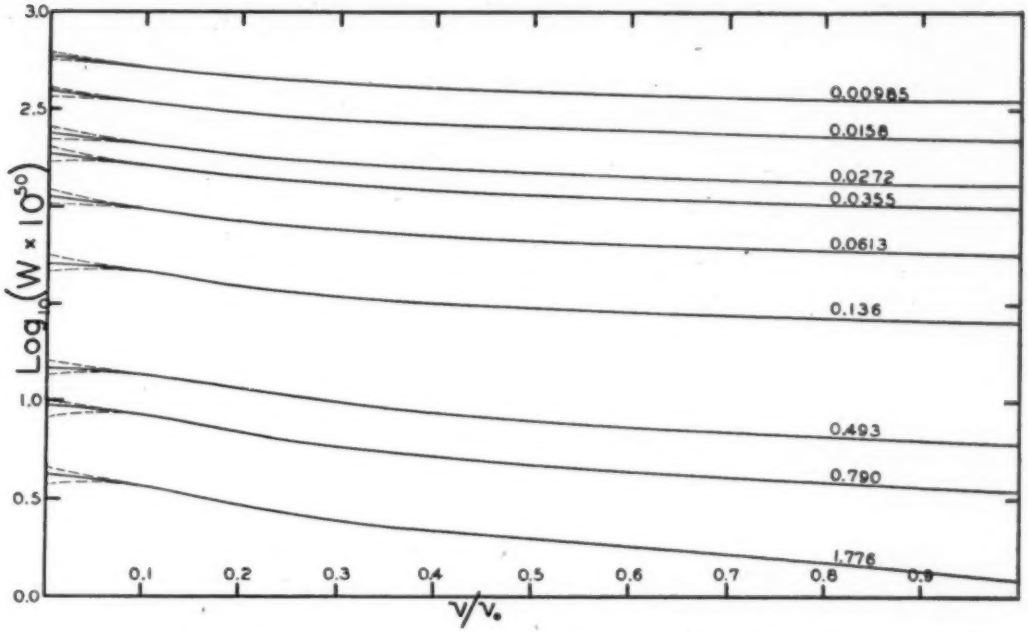

FIG. 7. Variation of total emitted energy W (in ergs per unit frequency interval per bombarding electron per atom-per-cm²) with position in the spectrum. The number immediately above each curve is the value of V/Z^2 to which the curve relates.

$I_{90°}=I_x+I_y$. Since the direction is a favorable one for observations, we show in Fig. 6 the predicted total intensities. The simplicity of these curves is remarkable. The variations of I_x and I_y with frequency, being opposite in sense, have nearly neutralized each other and given for this direction of observation a set of almost flat and linear spectra. It should be realized that these ordinates are proportional to intensities within *frequency* intervals of equal width. Most methods of spectrum observation seek to measure a quantity proportional to the intensities in *wave-length* bands of equal width, and such spectra are not flat and horizontal in character, though they may be readily transformed into a form comparable with the curves of Fig. 6.

Shifting the direction of observation to values of θ either greater or less than 90° tilts the curves of Fig. 6 so that their slight negative slopes become more accentuated and deepens their slight upward concavities.

The total radiation energy W, from Table II, is shown in Fig. 7.

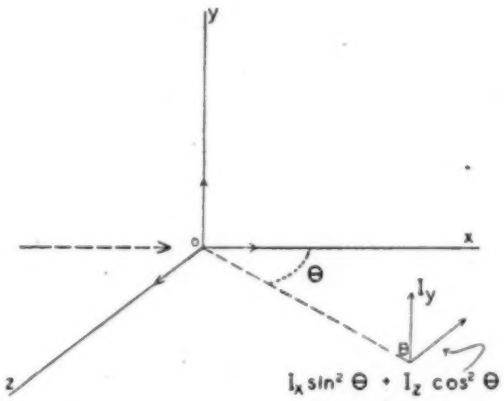

FIG. 8. Polarization components of radiation originating at 0 by bombardment from the left and observed at B.

POLARIZATION

In Fig. 8 the origin of radiation is O, the origin of the coordinate system, and the direction of observation is along OB, a direction specified by the angle θ. The two vectors erected at B illustrate the two components of radiation with mutually perpendicular directions of polarization. The component polarized with electric vector parallel to the y-axis of the coordinate system possesses intensity I_y; the component polarized in the perpendicular direction in general contains incoherent resolved contributions from both I_x and I_z and possesses the intensity $I_x \sin^2\theta + I_z \cos^2\theta$. We designate the ratio of these com-

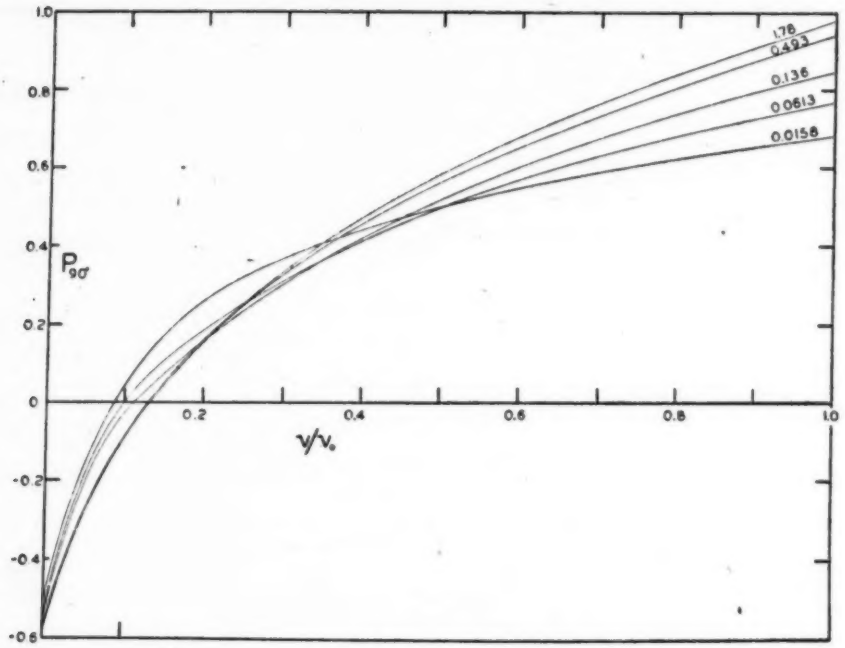

FIG. 9. Polarization of radiations emitted in a direction at 90° to the direction of bombardment. Numbers on the curves are values of V/Z^2. Scherzer investigated the case $V/Z^2=0$ and concluded that $P_{90°}=0.6$ for the entire spectrum. The curves above seem to be approaching this case as a reasonable limit.

ponents by $R=I_y/(I_z \sin^2\theta+I_x\cos^2\theta)$ and take as the measure of polarization the quantity $P_\theta=(1-R)/(1+R)$, an expression which is equivalent to $P_\theta=(1-D)/(1+D(2\csc^2\theta-1))$, where D is Sommerfeld's "depolarization ratio" I_y/I_x.

From the data of Table II, the polarization of radiation emitted in the direction $\theta=90°$ has been calculated for a number of values of ν/ν_0 and V/Z^2. These calculations are presented in the curves of Fig. 9. The similarity of these curves shows that in the present theory, polarization is more a matter of position in the spectrum than of atomic number or bombardment energy. Only in the vicinity of the short wave limit does $P_{90°}$ respond markedly to variations of V/Z^2. This response is in the sense that high V/Z^2 is associated with high (though in no case complete) polarization.

Near the low frequency end of the spectrum, radiation is produced at slight cost to the energy store of the bombardment electron. Close encounters are not required for such emission, and the predominant force component acting upon the electron lies in the yz-plane rather than the x-axis. The classical consequence, negative polarization, is supported by the curves of Fig. 9.

The curves of Fig. 9 present a somewhat tangled appearance, but they are in fact related in an entirely systematic manner, as a replot of the same data in Fig. 10 shows. The latter figure also emphasizes the relative insensitivity of polarization to V/Z^2 except in the region of low values of this quotient.

To show the theoretical variation of polarization with direction of emission, Fig. 11 is presented. Each curve here is a graph of P_θ *vs.* θ for the particular value of ν/ν_0 appended to the curve and for the value $V/Z^2=1.776$ common to all five curves. Polarization of the forward directed radiation is always zero; as θ increases, the polarization may increase or decrease, depending upon ν/ν_0. Evidently for a given V/Z^2, there is a value of ν/ν_0 for which the emitted radiation is substantially unpolarized for all directions of emission. The maximum (or minimum) of polarization at $\theta=90°$ is very flat in all cases, and observers in this angular region need have little concern about the precision of their θ measurements.

Alternative Methods of Calculation

Sauter's treatment of the problem made use of the first Born approximations. This method is

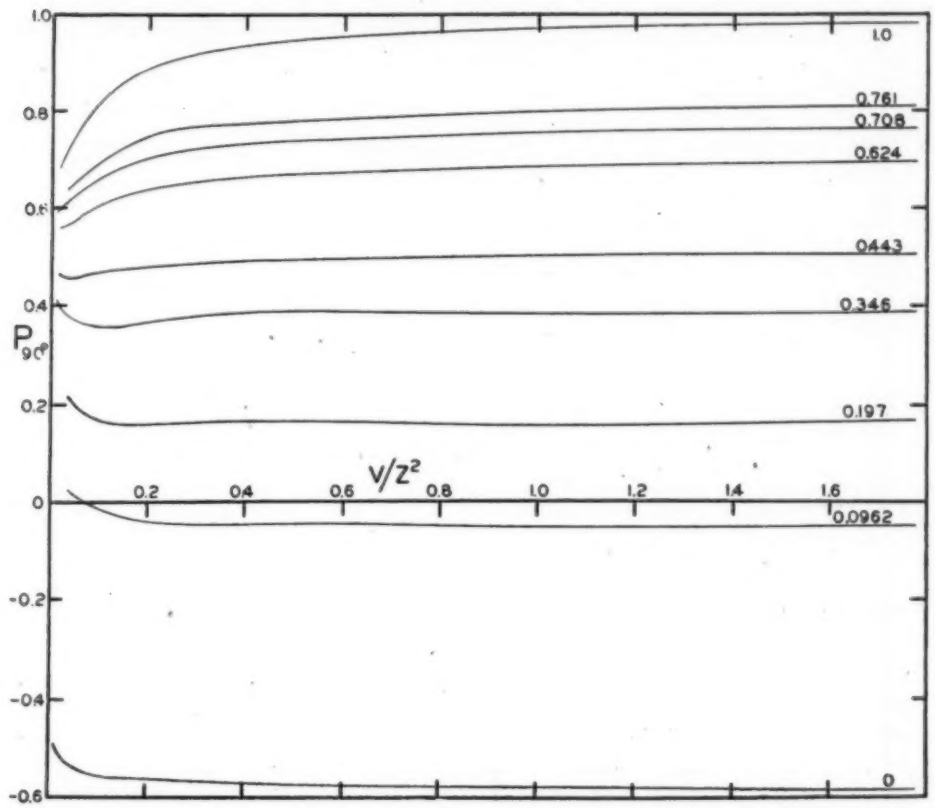

FIG. 10. Polarization of radiations emitted in a direction at 90° to the direction of bombardment. Numbers on the curves are ν/ν_0.

not appropriate unless $Ze^2/\hbar v_{1,2} \ll 1$, so the method is suited only to cases in which the bombardment electron moves with high speed both before and after collision. But such cases should properly be treated relativistically and with consideration of retardation of potential.

Sommerfeld and Maue made an important contribution to the solution of the difficult problem of integrating the absolute squares of the matrix elements over all directions of electron departure when they showed that the integration necessary to the determination of the *total radiation* in all directions could be carried out exactly. The integral contains a hypergeometric function which they evaluated by an approximation which has the effect of giving spectral intensity values which are somewhat too low, as Elwert later showed. Elwert's own method undoubtedly gives a better approximation, but it is not applicable at the short wave limit since it is restricted by the condition $(Ze^2/\hbar v_2)-(Ze^2/\hbar v_1)\ll 1$, $[(|n_2|-|n_1|)\ll 1$ in Elwert's notation] while at the long wave limit, it predicts infinite intensities exactly as does the Sauter treatment.

Upon finding that the total spectral energy given by his own development exceeded that of Sommerfeld and Maue by a factor

$$\left(\frac{\sinh|n_1|\pi}{|n_1|\pi}\right)^2$$

at the long wave limit and by almost the same factor at the other extreme, Elwert conjectured that the same factor might rectify the low intensity values of Sommerfeld and Maue in all cases. This supposition was strengthened by his investigation of its effect upon a single component of the matrix elements.

We have tested this supposition by comparing certain of our own calculations (in all of which the hypergeometric function in question has been evaluated without error in excess of one percent) with corresponding intensities derived from the Sommerfeld-Maue approximation rectified by

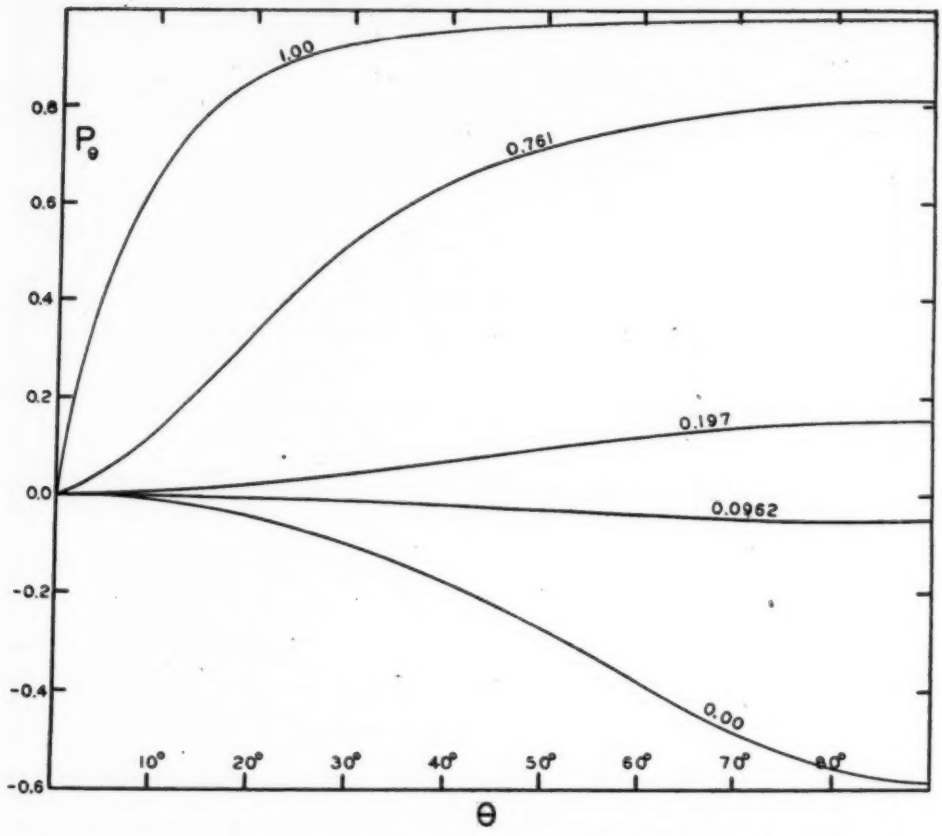

FIG. 11. Polarization *vs.* θ for $V/Z^2=1.776$ and for the values of ν/ν_0 shown on the curves. Since in the theory here considered P_θ is a function of sin θ, the curves for the next quadrant will be symmetrical continuations of those shown here.

Elwert's correction factor. Elwert's[12] statement of the Sommerfeld result is equivalent to

$$N_{sy}^2=\frac{4\pi n_1}{\xi_0^2 n_2}\{|n_1^2+n_2^2|\log(1-\xi_0)-4|n_1 n_2|\},$$

$$N_{sx}^2=\frac{8\pi n_1}{\xi_0^2 n_2}\{|n_1^2-n_2^2|\log(1-\xi_0)-4|n_1 n_2|\}.$$

The subscript s stands for Sommerfeld. Calculation of I_x, I_y, by these equations and the Elwert correction factor is exceedingly simple. Figure 12 shows that the resulting values are in good agreement with corresponding values from our Table II for the small values of $|n_2|-|n_1|$ involved in these cases. In Fig. 13 a similar comparison is made for larger values of $|n_1|-|n_2|$, and the agreement is much less good. Elwert's proposal for rectifying spectral intensities obtained under the Sommerfeld approximations is quite effective when his restricting assumption is satisfied.

[12] G. Elwert, reference 3, Eqs. (34) and (35).

Efficiency of Continuous X-Ray Production

From the radiation components of Table II, the efficiencies of continuous x-ray production in thin and thick targets may be computed. By thin target efficiency we mean the ratio of emitted x-ray energy (including all continuous spectrum frequencies and all directions of emission) to the total energy loss suffered by bombardment electrons in passing through a sample of matter so thin that only a small fraction of the electron energy is dissipated. This efficiency is not simply a function of V/Z^2, but of Z and V (or β) separately. The method of calculation in a given case is to select from Table II values of W pertaining to a common V/Z^2 but to different values of ν/ν_0 and to perform a numerical or

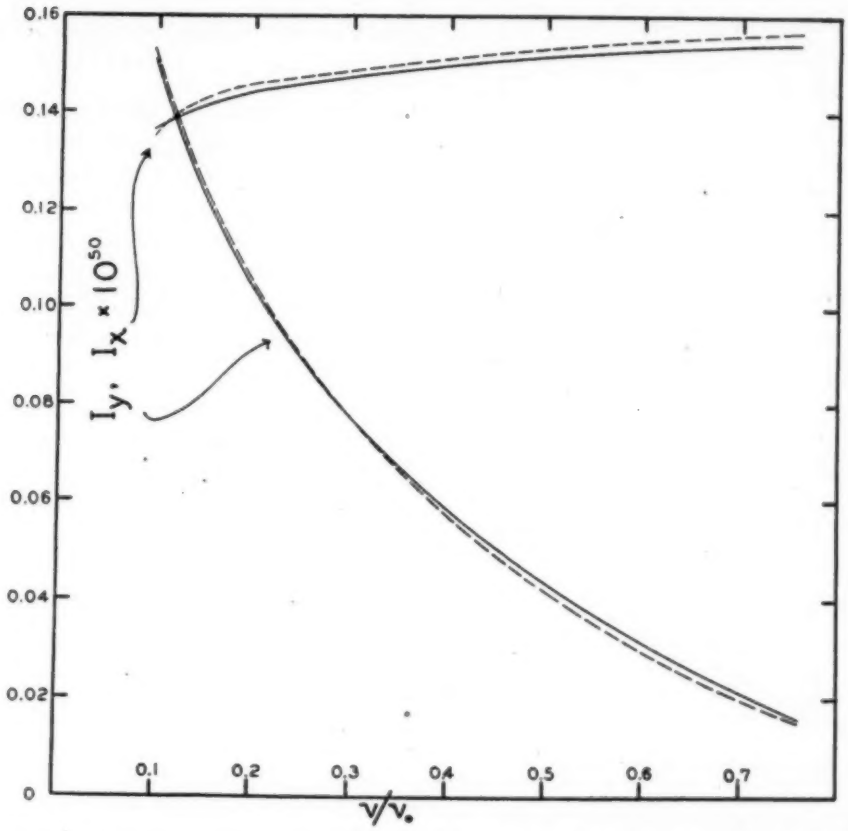

FIG. 12. Radiation components for the case $V/Z^2=1.776$. Solid curves from Table II. Broken curves by Elwert's method. Good agreement is found when $|n_2|-|n_1| \ll 1 \gg |n_1|$. For these curves $0.00829<|n_2|-|n_1|<0.167$, and $|n_1|=0.160$.

graphical integration yielding the total continuous x-ray energy per electron per atom-per-cm^2

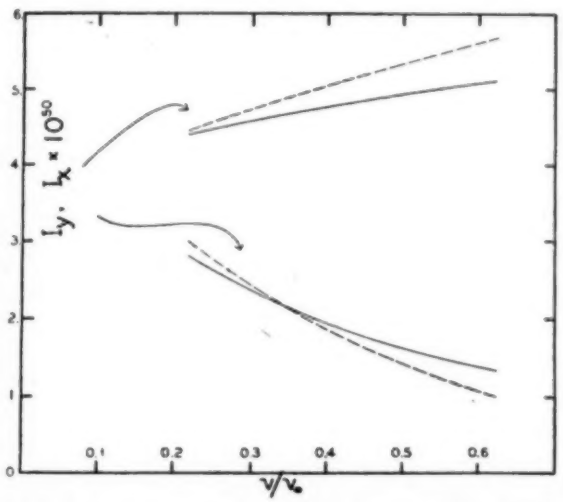

FIG. 13. Radiation components for the case $V/Z^2=0.0604$. Solid curves from Table II. Broken curves by Elwert's method, an approximation permissible only when $|n_1| \ll 1$ or $|n_2|-|n_1| \ll 1$. For these curves $0.112<|n_2|-|n_1|<0.545$, and $|n_1|=0.867$.

in all frequencies from zero to the quantum limit. It then remains to divide this result by the total energy lost per electron per atom-per-cm^2, a quantity depending upon β and Z.

In seeking the most probable values of this quantity, it seems best to adopt a combination of theoretical and experimental results. Williams[13] has shown that the observations of White[14] and Millington upon the stopping of fast β-particles by mica agree with the classical theory of Bohr[15] with respect to the functional variation of stopping power with β, though not as to the absolute values of stopping power. Williams recommends a practical formula consonant with these facts which may be put in the form:[16]

[13] E. J. Williams, Proc. Roy. Soc. **A130**, 310 (1931).
[14] P. White and G. Millington, Proc. Roy. Soc. **A120**, 701 (1928).
[15] N. Bohr, Phil. Mag. **25**, 10 (1913).
[16] An error in the statement of the energy-loss formula in Williams' Eq. (18) has been corrected above. We have also incorporated a suggestion of D. L. Webster that the right-hand side of Williams' formula be multiplied by $2Z/A$ where A is the atomic weight.

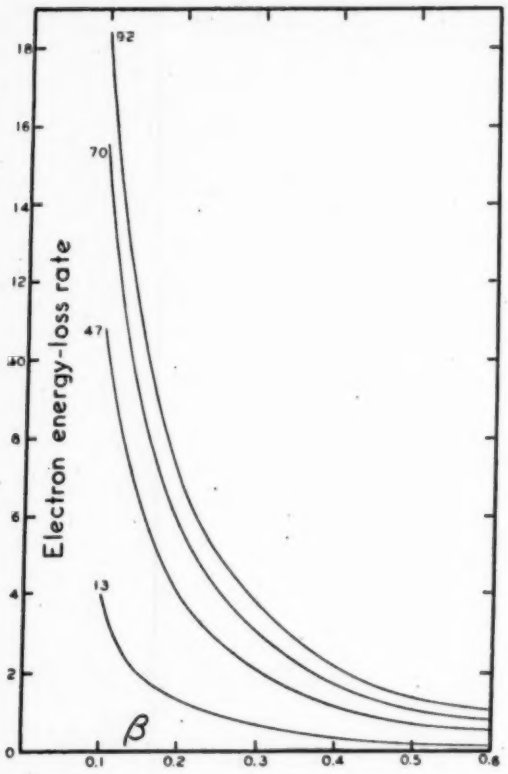

FIG. 14. Rates of loss of kinetic energy of electrons of speed β in the four substances indicated by atomic numbers near the curves. Ordinates are in ergs per atom-per-cm^2 $\times 10^{27}$ of stopping material. These curves are adapted from E. J. Williams' discussion of the theory of Bohr.

Energy loss in ergs per electron per atom-per-cm^2
$$= 5.62 Z\beta^{-1.4} \times 10^{-30}. \quad (10)$$

This formula takes no account of the losses by those few collisions in which an electron loses more than 1500 kev; its applicability above $\beta = 0.52$ has not been demonstrated nor has it been tested with elements of high Z. Nevertheless, we adopt it for all speeds above $\beta = 0.5$ and for all elements here considered.

For low values of β (up to $\beta = 0.30$) we draw upon Williams again. By cloud chamber observations of x-ray photoelectrons in oxygen, hydrogen, and argon, he established that in this energy region the Bohr theory predicts rates of energy loss about 60 percent too high, but correct in respect to their functional dependence upon Z and β. Assuming that the same ratio holds for all elements we multiply Bohr's expression by 0.6 and obtain:

Energy loss in ergs per electron per atom-per-cm^2
$$= 4.07 PZ\beta^{-2} \times 10^{-31}. \quad (11)$$

In this equation, P is a dimensionless function of Z and V involving the geometric mean of the natural frequencies of all electrons in the atom of the stopping material. P was calculated for $Z = 13$, 47, 70, and 92 by use of frequencies from the x-ray ν/R term values found in Siegbahn's *Spektroskopie der Röntgenstrahlen* (1931). Results of these calculations appear in Table IV.

To obtain electron energy losses in the region of energies intermediate between 20 and 80 kev, we have simply filled in by graphical interpolation in the manner illustrated by Fig. 14. It is gratifying that the calculations in the separated ranges admit of a smooth connecting curve segment for each of the four elements used in these efficiency calculations. We do not take this to mean that these curves are entirely correct; they may be wrong by many percent, but they would seem to be as good approximations as the literature of the subject affords at this date. Our efficiency conclusions will be subject to immediate amendment upon the appearance of more reliable energy-loss information. Figure 14, showing the energy-loss rates which we have adopted, may be used as a basis for any such adjustment.

TABLE IV. Electron-loss rates from Eq. (11).

Z	β	Kilovolts	P	Loss rate $\times 10^{28}$
13	0.10	2.57	7.47	39.5
13	0.15	5.84	8.38	19.7
13	0.20	10.53	9.97	13.2
13	0.25	16.75	10.77	9.12
13	0.30	24.66	11.43	6.72
47	0.10	2.57	5.66	108.2
47	0.15	5.84	7.12	61.0
47	0.20	10.53	8.15	39.0
47	0.25	16.75	8.96	27.4
47	0.30	24.66	9.61	20.4
70	0.10	2.57	5.46	155.5
70	0.15	5.84	6.92	87.6
70	0.20	10.53	7.95	56.7
70	0.25	16.75	8.76	39.9
70	0.30	24.66	9.41	29.8
92	0.10	2.57	4.91	183.8
92	0.15	5.84	6.42	106.8
92	0.20	10.53	7.40	69.3
92	0.25	16.75	8.21	49.2
92	0.30	24.66	8.86	36.9

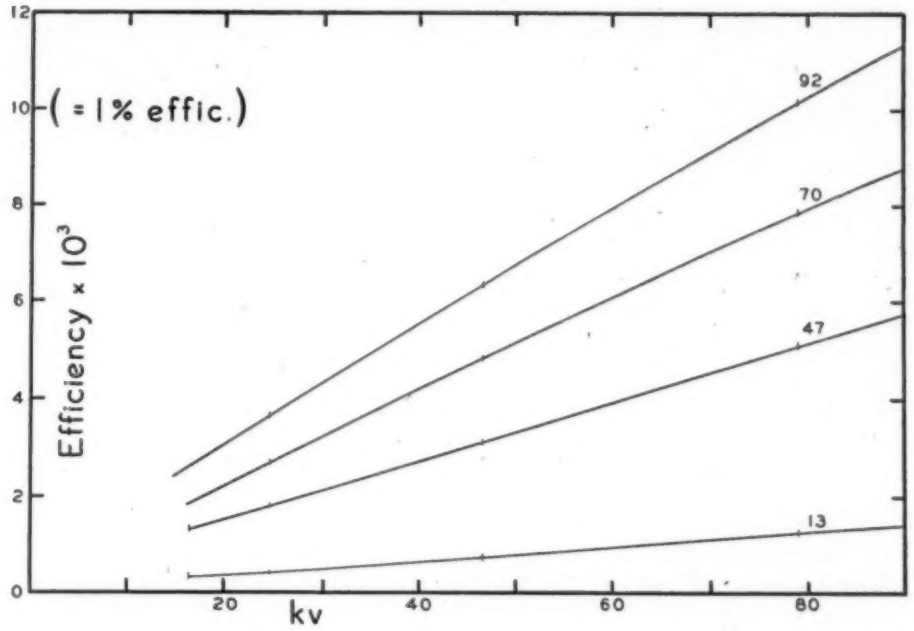

FIG. 15. Efficiency of production of continuous x-rays by thick targets of atomic numbers shown beside the curves when bombarded by electrons at the stated kv values. Curves are based on electron energy-loss rates of Fig. 14 and theoretical continuous spectrum energies derived from Table II. Approximate proportionality of efficiency to bombardment energy is shown.

The thin target efficiencies are given in Table V along with thick-target efficiencies now to be explained.

If E represents the varying kinetic energy of a typical electron slowing down from E_0 to zero in a thick target, then R, the output of x-ray energy, is given by

$$\int_0^{E_0} (dR/dE)dE,$$

where the derivative in parentheses is just the thin target efficiency. It is convenient to perform the graphical integration with respect to β, in which case we have

$$R=\int_0^{\beta_0} (dR/dE)(dE/d\beta)d\beta.$$

The second of the derivatives in the integrand is obtained by differentiating the relativistic expression for electron kinetic energy in terms of β. The integration has been carried out for twenty representative cases, and the over-all thick target efficiencies of continuous x-ray production have been obtained by dividing R by the initial electron energy. These efficiencies are shown in Table V, and Figs. 15 and 16.

From direct measurement of the x-ray and cathode ray energies Beatty[17] found thick target efficiencies to be proportional to the product of target atomic number and bombardment voltage. The many efficiency measurements made subsequently to Beatty's time were summarized by

TABLE V. Calculated efficiencies of continuous x-ray production in thin and thick targets of atomic number Z. For thin targets kev denotes the energy of bombardment at the point of production. For thick targets it denotes the energy at incidence upon the target.

kev	$Z=13$	$Z=47$	$Z=70$	$Z=92$	
15	5.3	23.0			thin target efficiencies $\times 10^4$
25	7.9	34.0	50.0	69.0	
40	13.0	53.0	80.0	110.0	
90	27.0	110.0	160.0	210.0	
16.8	3.4	14.0			thick target efficiencies $\times 10^4$
24.7	4.4	18.0	28.0	37.0	
46.5	7.6	32.0	48.0	64.0	
79.0	13.0	51.0	79.0	100.0	
90.0	15.0	58.0	88.0	110.0	

[17] R. T. Beatty, Proc. Roy. Soc. 89, 314 (1913).

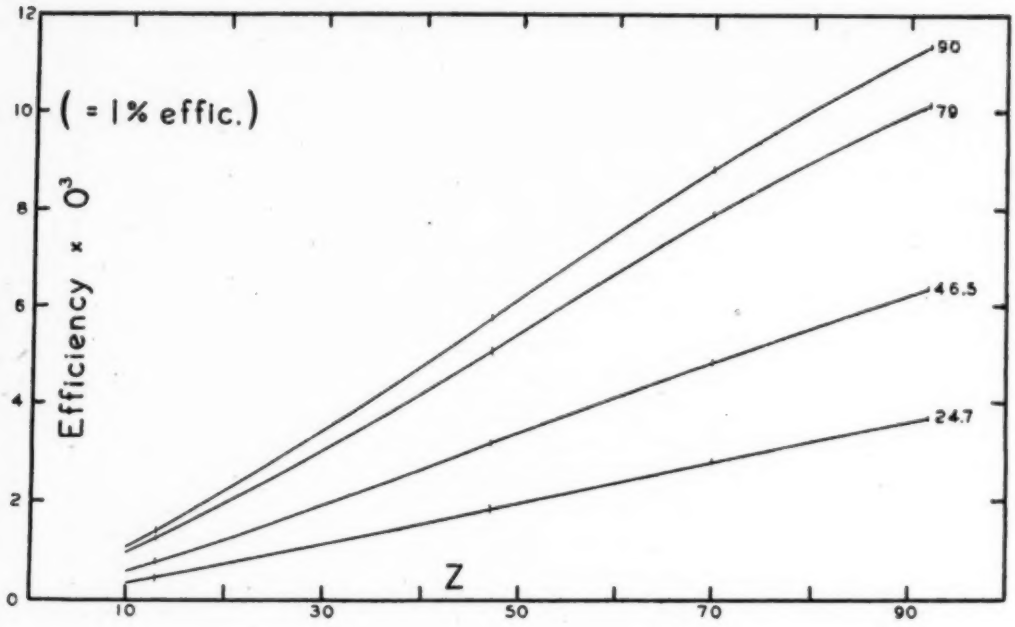

FIG. 16. Efficiency of production of continuous x-rays by thick targets of atomic numbers shown by the ordinate scale when bombarded by electrons at the kv values stated at the ends of the curves. Curves are based on electron energy-loss rates of Fig. 14 and theoretical continuous spectrum energies derived from Table II. Approximate proportionality of efficiency to atomic number is shown.

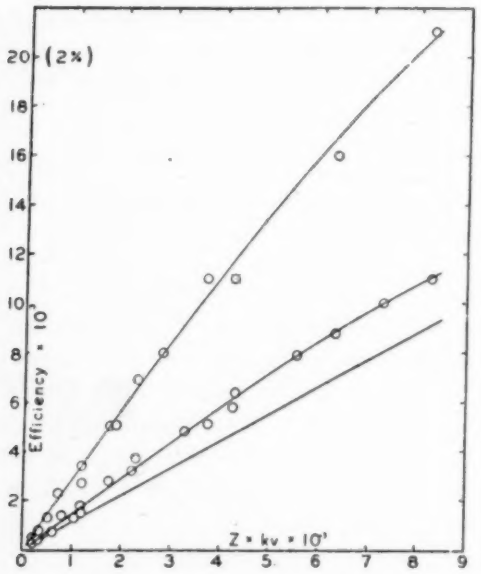

FIG. 17. Calculated efficiency of production of continuous x-rays in thin targets (upper curve) and thick targets (middle curve) of atomic number Z bombarded by electrons accelerated by potential kv. Approximate linearity of thick target curve supports Beatty's rule. Failure of points to lie precisely on the curves shows that efficiencies as here calculated are not strictly functions of the product Z kv. The lower curve is a consensus of the experimental results of several investigators of thick-target efficiency.

Compton and Allison[18] in the equation Efficiency $=1.1\times10^{-6}Z$ kv, a summary which these authors regarded as "probably correct within about 20 percent."

Figure 17 shows that the efficiencies of Table V[19] agree approximately with Beatty's rule and that they lie about 20 percent above the values of the consensus of Compton and Allison.

For any product Z kv the calculated thin target efficiency is about twice as great as the thick target efficiency. The thin target points in Fig. 17 are tolerably represented by the straight line

$$\text{Thin target efficiency} = 2.8\times10^{-6}Z \text{ kv}.$$

Calculated efficiencies for both thin and thick

[18] A. H. Compton and S. K. Allison, *X-Rays in Theory and Experiment* (D. Van Nostrand Company, Inc., New York, 1935), p. 90.

[19] The efficiencies of Table V lie somewhat below the values given earlier by one of the present authors (P. Kirkpatrick, Phys. Rev. **66**, 156 and 161 (1944)). The original calculations were based upon the energy-loss rates of Eq. (10) for low voltages as well as high. The values of W at $\nu=0$ in the previous work were obtained by graphical extrapolation instead of direct calculation. Changing the method of calculation in these two respects produced efficiencies more precisely proportional to Z and kv and, on the whole, lower.

targets tend to fall below linearity at high values of Z kv, but the reliability of the calculations is in question here since the potentials involved are outside the proper range of non-relativistic theory.

Dr. Robert Weinstock assisted the authors through numerous discussions in the early stages of the work, when his interpretation of his own papers contributed most helpfully to the development of the computational procedure.

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The Relativistic Correction in the Meson Theory of Nuclear Force

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In the present paper a systematic investigation is made on the relativistic corrections of the usual non-relativistic meson theories of nuclear force by expansion with respect to the dimensionless operator ∇/M. The mixed theory initiated by Møller and Rosenfeld is considered since it is the only satisfactory theory in the non-relativistic region. It is found that the inadmissible singularity which has been removed in the mixed theory in the non-relativistic region reappears in the higher approximations. This means that the extension of the mixed theory to the relativistic region as done by Møller and Rosenfeld in explaining the quadrupole moment is not justified.

I. INTRODUCTION

THE well known fact that the deuteron nucleus possesses an electric quadrupole moment is an indication of the existence of the tensor force in the nuclear interaction. The expression for this force derived from a single type of meson field, however, contains the inadmissible singularity of the type $1/r^3$, which has to be cut off at an arbitrary radius. It has been shown by Møller and Rosenfeld[1] that if the nuclear interaction is assumed as due to a mixed field of pseudoscalar and vector mesons of the same mass and suitably chosen coupling constants, then the tensor force disappears with its inadmissible singularity in the non-relativistic region. This provides a solution of the wave equation without resorting to the cut-off procedure. According to this theory, the quadrupole moment can only be a relativistic effect owing to the vanishing of the tensor force in the non-relativistic region. Møller and Rosenfeld have actually derived an expression for the quadrupole moment by considering the relativistic interaction of the first order. Schwinger[2] later suggested that if the mass of the vector meson is assumed to be larger than that of the pseudoscalar meson, then the tensor force reappears with an admissible singularity of the type $1/r$. This assumption is also in agreement with the hypothesis that the vector meson is highly unstable and responsible for the β-disintegration of the nucleus.

Recently Jauch and the present author[3] calculated numerically the quadrupole moment according to Schwinger's assumption, and found that the result is much smaller than the experimental value even for a very large mass ratio for these two types of mesons. This conclusion is quite independent of the coupling constants obtained from the scattering problems. Parallel to this calculation, Hulthén[4] also evaluated the quadrupole moment using the formula obtained by Møller and Rosenfeld. The

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[1] C. Møller and L. Rosenfeld, Kgl. Danske Vid. Sels. Math.-Fys. Medd **17** (1940).
[2] J. Schwinger, Phys. Rev. **61**, 287A (1942).
[3] J. M. Jauch and N. Hu, Phys. Rev. **65**, 289 (1944).
[4] L. Hulthén, Arkiv for Mat. Astr. Och. Fys. **29A**, No. 33 (1943).

result obtained by him is in reasonable agreement (judging from the error admissible by the approximation) with the experimental value. This would justify Møller and Rosenfeld's original view that the quadrupole moment is essentially a relativistic effect and therefore their explanation would be nearer to the fact than Schwinger's explanation.

The original derivation of the formula for the quadruple moment by Møller and Rosenfeld is, however, not altogether consistent. In its derivation, the relativistic force is added as a perturbation term to the unperturbed static wave equation without considering the other terms of the same order of magnitude arising from the non-relativistic approximation. In the present investigation we retain these terms consistently and find that this non-static force gives rise to a tensor force also of the inadmissible type $1/r^3$, which in turn requires further adjustment of the coupling constants to make it vanish. Consequently, the relativistic effect, if made regular, cannot make any contribution to the quadruple moment.

It should be noted that the inadmissible singularity of the type $1/r^2$ also exists in the equations used by Møller and Rosenfeld. No serious attention, however, has been paid to this since the first perturbation leads to convergent and unique results. Application of the perturbation method in our case also gives convergent and unique results in the first perturbation. These results can only be considered as provisional and further investigation is needed to find out their significance, as we know that the equations give no stationary states when solved rigorously unless additional assumptions equivalent to a cut-off are introduced.

II. THE NON-RELATIVISTIC APPROXIMATION ON THE DEUTERON WAVE EQUATION

The rigorous relativistic interaction potential obtained by Kemmer,[5] Bhabha,[6] and Møller-Rosenfeld[1] for a mixture of pseudoscalar and vector mesons is given by

$$V=F+W+K,$$

$$F=(\tau^{(1)}\tau^{(2)})\left[g_1{}^2+g_2{}^2(\sigma^{(1)}\sigma^{(2)})-\frac{1}{\mu^2}(f_2{}^2-g_2{}^2\rho_3{}^{(1)}\rho_3{}^{(2)})(\sigma^{(1)}\nabla^{(1)})(\sigma^{(2)}\nabla^{(2)})\right]\tilde{\omega}(r),$$

$$W=(\tau^{(1)}\tau^{(2)})\left\{\frac{f_1f_2}{\mu}(\rho^{(1)}\sigma^{(2)}-\rho^{(2)}\sigma^{(1)})-\frac{g_1g_2}{\mu}[\rho_2{}^{(1)}\sigma^{(1)}-\rho_2{}^{(2)}\sigma^{(2)}\right.$$
$$\left.+(\rho_1{}^{(1)}\rho_3{}^{(2)}+\rho_1{}^{(2)}\rho_3{}^{(1)})(\sigma^{(1)}\times\sigma^{(2)})]\right\}\nabla^{(1)}\tilde{\omega}(r), \tag{1}$$

$$K=(\tau^{(1)}\tau^{(2)})\left\{-f_1{}^2\rho_3{}^{(1)}\rho_3{}^{(2)}\rho_1{}^{(1)}\rho_1{}^{(2)}-g_1{}^2\rho_1{}^{(1)}\rho_1{}^{(2)}(\sigma^{(1)}\sigma^{(2)})\right.$$
$$\left.-\frac{g_1{}^2}{\mu^2}\rho_1{}^{(1)}\rho_1{}^{(2)}(\sigma^{(1)}\nabla^{(1)})(\sigma^{(2)}\nabla^{(2)})-\frac{g_2{}^2}{\mu^2}\rho_1{}^{(1)}\rho_1{}^{(2)}\rho_3{}^{(1)}\rho_3{}^{(2)}(\sigma^{(1)}\nabla^{(1)})(\sigma^{(2)}\nabla^{(2)})\right\}\tilde{\omega}(r),$$

$$\tilde{\omega}(r)=\frac{e^{-\mu r}}{4\pi r},\quad \nabla^{(1)}=-\nabla^{(2)}=\nabla,$$

where F gives the static interaction. W and K are the non-static interactions linear and bilinear in $\rho_1{}^{(\nu)}$ and $\rho_2{}^{(\nu)}$ $(\nu=1, 2)$, respectively. μ is the mass of the mesons (we are now considering the case that the pseudoscalar and the vector mesons have the same mass), ρ_1, ρ_2, ρ_3 and σ are the usual Dirac matrices. $(\nu)=1, 2$ refers to the first and the second nucleus respectively, and $\tau^{(1)}$ and $\tau^{(2)}$ are the isotopic spin matrices. f_1 and f_2 are, respectively, the spin-independent and the spin-dependent

[5] N. Kemmer, Proc. Roy. Soc. **166**, 127 (1938).
[6] H. J. Bhabha, Proc. Roy. Soc. **166**, 501 (1938).

coupling constants for the pseudoscalar meson, and g_1 and g_2 are the same constants[7] for the vector meson. The unit $c=\hbar=1$ is used in the present paper.

The relativistic wave equation is then

$$\left\{-\frac{1}{i}\frac{\partial}{\partial t}+\frac{1}{i}(\rho_1{}^{(1)}\boldsymbol{\sigma}^{(1)}\nabla^{(1)}+\rho_1{}^{(2)}\boldsymbol{\sigma}^{(2)}\nabla^{(2)})+M(\rho_3{}^{(1)}+\rho_3{}^{(2)})+V\right\}\psi(x^{(1)}, x^{(2)})=0, \tag{2}$$

where V is given by (1). ψ is the spinor with sixteen components formed from the direct product of the two four-component spinor representations of the two nucleons. We may conveniently represent ψ by $\psi_{\alpha\beta}$ ($\alpha, \beta=1, 2, 3, 4$), where the first index goes only with the operators referred to the nucleon (1) and the second index goes only with those referred to the nucleon (2).

It is seen from (1) that the tensor force of the form $(\boldsymbol{\sigma}^{(1)}\nabla^{(1)})(\boldsymbol{\sigma}^{(2)}\nabla^{(2)})\bar{\omega}(r)$ gives rise to an inadmissible singularity of the form $1/r^3$ which gives no stationary solution for the deuteron wave equation. If we put $g_1{}^2=g_2{}^2$, then we get the well-known Møller-Rosenfeld mixture of which the tensor force with its inadmissible singularity disappears entirely from the potential F in the non-relativistic region. It should be remembered that Møller and Rosenfeld's theory was originally proposed for the static region only. In the relativistic region, as we can easily see from (1), the inadmissible singularity of the type $1/r^3$ still remains. Therefore it is meaningless to solve the relativistic equation (2) rigorously for the Møller-Rosenfeld mixture, as the latter gives no better service than the other simple theories. The situation will be different, however, if we try to solve (2) by the non-relativistic approximation of which the first approximation is the static case. Then it is quite justified to start with the mixed theory of Møller and Rosenfeld which alone gives regular singularity in the first approximation. The second approximation would correspond to the region in which the Møller-Rosenfeld expression for the quadrupole moment was derived.

To carry out the approximation, we follow the well known procedure[8] of expanding ψ in powers of the dimensionless operator ∇/M where $1/M$ is the Compton wave-length of the nucleon. We put

$$\begin{aligned}
(\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}) &= (\varphi_1, \varphi_2, \varphi_3, \varphi_4)e^{iMt},\\
(\psi_{31}, \psi_{32}, \psi_{41}, \psi_{42}) &= (\chi_1{}^{(1)}, \chi_2{}^{(1)}, \chi_3{}^{(1)}, \chi_4{}^{(1)})e^{iMt},\\
(\psi_{13}, \psi_{14}, \psi_{23}, \psi_{24}) &= (\chi_1{}^{(2)}, \chi_2{}^{(2)}, \chi_3{}^{(2)}, \chi_4{}^{(2)})e^{iMt},\\
(\psi_{33}, \psi_{34}, \psi_{43}, \psi_{44}) &= (\phi_1, \phi_2, \phi_3, \phi_4)e^{iMt}.
\end{aligned} \tag{3}$$

Substituting (3) into (2), we obtain

$$-\frac{1}{i}\frac{\partial\varphi}{\partial t}+\frac{1}{i}(\boldsymbol{\sigma}^{(1)}\nabla^{(1)})\chi^{(1)}+\frac{1}{i}(\boldsymbol{\sigma}^{(2)}\nabla^{(2)})\chi^{(2)}+V_{\varphi\varphi}\varphi+V_{\varphi\chi^{(1)}}\chi^{(1)}+V_{\varphi\chi^{(2)}}\chi^{(2)}+V_{\varphi\phi}\phi=0, \tag{4a}$$

$$-\frac{1}{i}\frac{\partial\chi^{(1)}}{\partial t}+\frac{1}{i}(\boldsymbol{\sigma}^{(1)}\nabla^{(1)})\varphi+\frac{1}{i}(\boldsymbol{\sigma}^{(2)}\nabla^{(2)})\phi-2M\chi^{(1)}$$
$$+V_{\chi^{(1)}\varphi}\varphi+V_{\chi^{(1)}\chi^{(1)}}\chi^{(1)}+V_{\chi^{(1)}\chi^{(2)}}\chi^{(2)}+V_{\chi^{(1)}\phi}\phi=0, \tag{4b}$$

$$-\frac{1}{i}\frac{\partial\chi^{(2)}}{\partial t}+\frac{1}{i}(\boldsymbol{\sigma}^{(1)}\nabla^{(1)})\phi+\frac{1}{i}(\boldsymbol{\sigma}^{(2)}\nabla^{(2)})\varphi-2M\chi^{(2)}$$
$$+V_{\chi^{(2)}\varphi}\varphi+V_{\chi^{(2)}\chi^{(1)}}\chi^{(1)}+V_{\chi^{(2)}\chi^{(2)}}\chi^{(2)}+V_{\chi^{(2)}\phi}\phi=0, \tag{4c}$$

$$-\frac{1}{i}\frac{\partial\phi}{\partial t}+\frac{1}{i}(\boldsymbol{\sigma}^{(1)}\nabla^{(1)})\chi^{(2)}+\frac{1}{i}(\boldsymbol{\sigma}^{(2)}\nabla^{(2)})\chi^{(1)}-4M\phi+V_{\phi\varphi}\varphi+V_{\phi\chi^{(1)}}\chi^{(1)}+V_{\phi\chi^{(2)}}\chi^{(2)}+V_{\phi\phi}\phi=0, \tag{4d}$$

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[7] The nomenclature that g_1 is the spin-independent coupling constant is only justified in the non-relativistic region. In the relativistic region, as we can see from (1), the spin-dependent interaction terms also contain g_1.

[8] W. Pauli, *Handbuch der Physik* (Verlagsbuchhandlung, Julius Springer, Berlin, 1933), Vol. 24, Pt. 1, p. 237.

where φ, $\chi^{(1)}$, $\chi^{(2)}$, ϕ are the four-component wave functions defined by (3). $\sigma^{(1)}$ and $\sigma^{(2)}$ are respectively the direct products $\sigma\times I$ and $I\times\sigma$, σ being the usual Pauli matrices and $I=\begin{pmatrix}10\\01\end{pmatrix}$. $V_{\chi^{(1)}\varphi}$ is the submatrix of V that transforms a φ into a $\chi^{(1)}$ and $V_{\chi^{(1)}\chi^{(1)}}$, $V_{\chi^{(1)}\chi^{(2)}}$, $\cdots$ have similar meanings. We can easily verify from (1) that

$$\begin{aligned}
V_{\varphi\varphi}&=V_{\chi^{(1)}\chi^{(1)}}=V_{\chi^{(2)}\chi^{(2)}}=V_{\phi\phi}=F_1,\\
V_{\varphi\chi^{(1)}}&=V_{\chi^{(1)}\varphi}=V_{\chi^{(2)}\phi}=V_{\phi\chi^{(2)}}=G_1,\\
V_{\varphi\chi^{(2)}}&=V_{\chi^{(2)}\varphi}=V_{\chi^{(1)}\phi}=V_{\phi\chi^{(1)}}=H_1,\\
V_{\varphi\phi}&=V_{\phi\varphi}=V_{\chi^{(1)}\chi^{(2)}}=V_{\chi^{(2)}\chi^{(1)}}=K_1,
\end{aligned}\tag{5}$$

where

$$\begin{aligned}
F_1&=(\tau^{(1)}\tau^{(2)})\left[g_1^2+g_2^2(\sigma^{(1)}\sigma^{(2)})-\frac{1}{\mu^2}(f_2^2-g_2^2\epsilon_3^{(1)}\epsilon_3^{(2)})(\sigma^{(1)}\nabla^{(1)})(\sigma^{(2)}\nabla^{(2)})\right]\tilde{\omega}(r),\\
G_1&=(\tau^{(1)}\tau^{(2)})\left\{\frac{f_1f_2}{\mu}\epsilon_2^{(1)}\sigma^{(2)}-\frac{g_1g_2}{\mu}[\epsilon_2^{(1)}\sigma^{(1)}+\epsilon_3^{(2)}(\sigma^{(1)}\times\sigma^{(2)})]\right\}\cdot\nabla^{(1)}\tilde{\omega}(r),\\
H_1&=(\tau^{(1)}\tau^{(2)})\left\{\frac{f_1f_2}{\mu}\epsilon_2^{(2)}\sigma^{(1)}-\frac{g_1g_2}{\mu}[\epsilon_2^{(2)}\sigma^{(2)}+\epsilon_3^{(1)}(\sigma^{(2)}\times\sigma^{(1)})]\right\}\cdot\nabla^{(2)}\tilde{\omega}(r),\\
K_1&=(\tau^{(1)}\tau^{(2)})\left\{-f_1^2\epsilon_3^{(1)}\epsilon_3^{(2)}-g_1^2(\sigma^{(1)}\sigma^{(2)})+\frac{g_1^2}{\mu^2}(\sigma^{(1)}\nabla)(\sigma^{(2)}\nabla)+\frac{g_2^2}{\mu^2}\epsilon_3^{(1)}\epsilon_3^{(2)}(\sigma^{(1)}\nabla)(\sigma^{(2)}\nabla)\right\}\tilde{\omega}(r),
\end{aligned}\tag{6}$$

where $f_2^2=g_2^2$ since we are now considering the Møller-Rosenfeld mixture. The quantities $\epsilon_\kappa^{(\nu)}$ are defined in Table I, where the first column gives the value of $\epsilon_\kappa^{(\nu)}$ when F_1, G_1, H_1, or K_1 is applied to φ, and similar meanings for the the other columns.

Before introducing the non-relativistic approximation, we shall first estimate the order of magnitude of the different terms in (4) with respect to the dimensionless operator ∇/M. The expectation value of the operator ∇ is roughly of the order $1/r_0$, where r_0 is the radius of the deuteron nucleus. It follows that ∇/μ is of the order unity since $\mu\sim r_0^{-1}$. On the other hand ∇/M is a quantity much smaller than unity (roughly of the order $\mu/M\sim 0.1$). Therefore ∇/M is a suitable parameter for a perturbation approximation. Furthermore, in the derivation of (1) a perturbation expansion with respect to the parameters $g_\alpha/(4\pi)^{\frac{1}{2}}$ and $f_\alpha/(4\pi)^{\frac{1}{2}}$ ($\alpha=1, 2$) has already been used. The order of magnitude of $g_\alpha^2/4\pi$ and $f_\alpha^2/4\pi$ determined from the deuteron problem is also found to be very small (roughly ~ 0.1). Therefore in the following approximation we shall also count $g_\alpha^2/4\pi$ and $f_\alpha^2/4\pi$ as small quantities of the first order. In other words, whenever terms of the nth order in ∇/M are neglected in the part proportional to g_α^2, f_α^2, or $f_\alpha g_\alpha$, only terms of $(n+1)$th order are neglected in the part independent of the interaction. In the following we shall refer the order of magnitude only to the interaction part of our equations. The present method of expanding ψ's in powers of ∇/M is essentially the non-relativistic approximation since ∇/M applied to φ gives the velocity of the nucleon measured in the unit $c=1$. We shall see later that the usual non-relativistic wave equation can only be obtained when terms of the order ∇/M are neglected in (4).

Since only (4a) of (4) does not contain the term of the form M we may consider φ as zeroth order of magnitude. We can easily verify from (4b, c, d) with the help of (5) and (6) that $\chi^{(1)}$ and $\chi^{(2)}$ are of the order ∇/M, and ϕ is of the

TABLE I.

	φ	$\chi^{(1)}$	$\chi^{(2)}$	ϕ
$\epsilon_3^{(1)}$	1	-1	1	-1
$\epsilon_3^{(2)}$	1	1	-1	-1
$\epsilon_2^{(1)}$	i	$-i$	i	$-i$
$\epsilon_2^{(2)}$	i	i	$-i$	$-i$

order ∇^2/M^2. Neglecting terms of the order ∇^2/M^2 in (4a) and of the order ∇^3/M^3 in (4b, c, d), we obtain

$$-\frac{1}{i}\frac{\partial\varphi}{\partial t}+\frac{1}{i}(\sigma^{(1)}\nabla^{(1)})\chi^{(1)}+\frac{1}{i}(\sigma^{(2)}\nabla^{(2)})\chi^{(2)}+V_{\varphi\varphi}\varphi+V_{\varphi\chi^{(1)}}\chi^{(1)}+V_{\varphi\chi^{(2)}}\chi^{(2)}=0, \tag{7a}$$

$$2M\chi^{(1)}=\frac{1}{i}(\sigma^{(1)}\nabla^{(1)})\varphi+V_{\chi^{(1)}\varphi}\varphi, \tag{7b}$$

$$2M\chi^{(2)}=\frac{1}{i}(\sigma^{(2)}\nabla^{(2)})\varphi+V_{\chi^{(2)}\varphi}\varphi, \tag{7c}$$

$$4M\phi=V_{\phi\varphi}\varphi. \tag{7d}$$

Substituting (7b) and (7c) into (7a), we obtain

$$-\frac{1}{i}\frac{\partial\varphi}{\partial t}-\frac{1}{2M}[\nabla^{(1)^2}+\nabla^{(2)^2}]\varphi+V_{\varphi\varphi}\varphi+\frac{1}{2Mi}\{(\sigma^{(1)}\nabla^{(1)})(V_{\chi^{(1)}\varphi}\varphi)+V_{\varphi\chi^{(1)}}(\sigma^{(1)}\nabla^{(1)})\varphi$$
$$+(\sigma^{(2)}\nabla^{(2)})(V_{\chi^{(2)}\varphi}\varphi)+V_{\varphi\chi^{(2)}}(\sigma^{(2)}\nabla^{(2)})\varphi\}=0, \tag{8}$$

where from (6)

$$V_{\varphi\varphi}=(\tau^{(1)}\tau^{(2)})[g_1^2+g_2^2(\sigma^{(1)}\sigma^{(2)})]\bar{\omega}(r),$$
$$V_{\varphi\chi^{(1)}}=(\tau^{(1)}\tau^{(2)})\left\{-\frac{f_1f_2}{\mu}i\sigma^{(2)}+\frac{g_1g_2}{\mu}i\sigma^{(1)}-\frac{g_1g_2}{\mu}(\sigma^{(1)}\times\sigma^{(2)})\right\}\cdot\nabla^{(1)}\bar{\omega}(r),$$
$$V_{\chi^{(1)}\varphi}=(\tau^{(1)}\tau^{(2)})\left\{\frac{f_1f_2}{\mu}i\sigma^{(2)}-\frac{g_1g_2}{\mu}i\sigma^{(1)}-\frac{g_1g_2}{\mu}(\sigma^{(1)}\times\sigma^{(2)})\right\}\cdot\nabla^{(1)}\bar{\omega}(r), \tag{9}$$
$$V_{\varphi\chi^{(2)}}=(\tau^{(1)}\tau^{(2)})\left\{-\frac{f_1f_2}{\mu}i\sigma^{(1)}+\frac{g_1g_2}{\mu}i\sigma^{(2)}-\frac{g_1g_2}{\mu}(\sigma^{(2)}\times\sigma^{(1)})\right\}\cdot\nabla^{(2)}\bar{\omega}(r),$$
$$V_{\chi^{(2)}\varphi}=(\tau^{(1)}\tau^{(2)})\left\{\frac{f_1f_2}{\mu}i\sigma^{(1)}-\frac{g_1g_2}{\mu}i\sigma^{(2)}-\frac{g_1g_2}{\mu}(\sigma^{(2)}\times\sigma^{(1)})\right\}\cdot\nabla^{(2)}\bar{\omega}(r).$$

If we further neglect the terms of the order ∇/M in (8), then we have

$$-\frac{1}{i}\frac{\partial\varphi}{\partial t}-\frac{1}{2M}[\nabla^{(1)^2}+\nabla^{(2)^2}]\varphi+V_{\varphi\varphi}\varphi=0, \tag{10}$$

which is just the non-relativistic wave equation for the deuteron with the static interaction potential $V_{\varphi\varphi}$ of Møller and Rosenfeld. It should be noted that $[(1/M^n)\nabla^n\bar{\omega}(r)]$ is no longer a small quantity of the order n when r is very small, since $\bar{\omega}(r)$ behaves as $1/r$ for small r. Consequently the neglect of the terms of the form $[(1/M^n)\nabla^n\bar{\omega}(r)]\varphi$ in passing from (4) to (8) and (10) is justified only when φ is also very small for small r, so that the contribution of these terms is still negligible. This fact can be utilized as a test of the validity of our approximation, since φ can be determined completely from the wave equations. Physically this means that the two nucleons must always remain sufficiently apart from each other in order that the present non-relativistic approximation can be used.

Equation (8) is the approximate wave equation for the deuteron accurate up to the first order in ∇/M. Substituting (9) into (8) we obtain after some calculation

$$-\frac{1}{i}\frac{\partial\varphi}{\partial t}-\frac{1}{2M}[\nabla^{(1)^2}+\nabla^{(2)^2}]\varphi+V_{\varphi\varphi}\varphi+P\varphi=0, \tag{11}$$

with

$$P\varphi=-\frac{1}{M\mu}(\tau^{(1)}\tau^{(2)})(f_1f_2+g_1g_2)[(\sigma^{(1)}\nabla^{(1)})(\sigma^{(2)}\nabla^{(2)})\tilde{\omega}]\cdot\varphi$$
$$+\frac{1}{M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2[(\sigma^{(1)}+\sigma^{(2)},\ \nabla^{(1)}\tilde{\omega}\times\nabla^{(1)}\varphi)+(\sigma^{(1)}+\sigma^{(2)},\ \nabla^{(2)}\tilde{\omega}\times\nabla^{(2)}\varphi)]$$
$$-\frac{1}{M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2[1+(\sigma^{(1)}\sigma^{(2)})]\mu^2\tilde{\omega}\cdot\varphi, \quad (12)$$

where we have replaced $\nabla^2\tilde{\omega}$ by $\mu^2\tilde{\omega}$ in the third term of (12), since $\nabla^2\tilde{\omega}-\mu^2\tilde{\omega}=\delta(r)$, the delta function in the interaction being neglected as usual. It can easily be recognized that the singularity of the first term of (12) is of the inadmissible type $1/r^3$. To investigate the nature of the singularity of the second term, we put it in a slightly different form; using the center-of-mass coordinate system of the deuteron in which $\nabla^{(1)}\varphi=-\nabla^{(2)}\varphi$,

$$\frac{1}{M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2[(\sigma^{(1)}+\sigma^{(2)},\ \nabla^{(1)}\tilde{\omega}\times\nabla^{(1)}\varphi)+(\sigma^{(1)}+\sigma^{(2)},\ \nabla^{(2)}\tilde{\omega}\times\nabla^{(2)}\varphi)]$$
$$=\frac{2}{M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2\frac{d\tilde{\omega}}{dr}\cdot\left(\sigma^{(1)}+\sigma^{(2)},\ \frac{\mathbf{r}}{r}\times\nabla\varphi\right)=\frac{2}{M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2\frac{1}{r}\frac{d\tilde{\omega}}{dr}\cdot(\sigma^{(1)}+\sigma^{(2)},\ \mathbf{L}\varphi), \quad (13)$$

where $\mathbf{L}$ is the angular momentum operator defined by

$$L_x=y(\partial/\partial z)-z(\partial/\partial y),\quad L_y=z(\partial/\partial x)-x(\partial/\partial z),\quad L_z=x(\partial/\partial y)-y(\partial/\partial x). \quad (14)$$

It has been shown by Kemmer[9] that the form of φ for the ground (triplet) state of the deuteron is

$$\varphi=A\varphi_{l=0}+B\varphi_{l=2}, \quad (15)$$

$$\varphi_{l=0}\equiv\begin{pmatrix}\psi_{11}^{(0)} & \psi_{12}^{(0)}\\ \psi_{21}^{(0)} & \psi_{22}^{(0)}\end{pmatrix}=\mathcal{Y}_0^m\frac{R_0(r)}{r}\equiv\begin{pmatrix}[m(m+1)]^{\frac{1}{2}}Y_0^{m-1}, & -[(m+1)(1-m)]^{\frac{1}{2}}Y_0^m\\ -[(1+m)(1-m)]^{\frac{1}{2}}Y_0^m, & [m(m-1)]^{\frac{1}{2}}Y_0^{m+1}\end{pmatrix}\frac{1}{\sqrt{2}}\frac{R_0(r)}{r}, \quad (16)$$

$$\varphi_{l=2}\equiv\begin{pmatrix}\psi_{11}^{(2)} & \psi_{12}^{(2)}\\ \psi_{21}^{(2)} & \psi_{22}^{(2)}\end{pmatrix}=\mathcal{Y}_2^m\frac{R_2(r)}{r}\equiv\begin{pmatrix}[(2-m)(3-m)]^{\frac{1}{2}}Y_2^{m-1}, & [(2+m)(2-m)]^{\frac{1}{2}}Y_2^m\\ [(2+m)(2-m)]^{\frac{1}{2}}Y_2^m, & [(2+m)(3+m)]^{\frac{1}{2}}Y_2^{m+1}\end{pmatrix}\frac{1}{2\sqrt{5}}\frac{R_2(r)}{r}, \quad (17)$$

where A and B are two arbitrary constants satisfying $A^2+B^2=1$, l is the orbital quantum number and $m=0,\ \pm1$ are the quantum numbers for the component of the spin in the z-direction. Y_l^m is the normalized spherical harmonics, $\mathcal{Y}_0^m$ and $\mathcal{Y}_2^m$ stand for the matrices in (16) and (17) respectively, and R_1 and R_2 are two normalized radial functions. Substituting (15)–(17) into (13) and noticing hat $\mathbf{L}\varphi_{l=0}=0$, we find after some reduction

$$\frac{1}{M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2[(\sigma^{(1)}+\sigma^{(2)},\ \nabla^{(1)}\tilde{\omega}\times\nabla^{(1)}\varphi)+(\sigma^{(1)}+\sigma^{(2)},\ \nabla^{(2)}\tilde{\omega}\times\nabla^{(2)}\varphi)]$$
$$=-\frac{12}{M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2\frac{1}{r}\frac{d\tilde{\omega}}{dr}\cdot B\varphi_{l=2}=\frac{12}{4\pi M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2\left(\frac{1}{r^3}+\frac{\mu}{r^2}\right)e^{-\mu r}\cdot B\varphi_{l=2}. \quad (18)$$

Therefore the second term of the right-hand side of (12) also gives rise to a singularity of the inad-

[9] N. Kemmer, Helv. Phys. Acta **10**, 47 (1937).

missible type $1/r^3$. After some reduction on the first term, (12) becomes

$$P\varphi=\frac{1}{4\pi M\mu}(\tau^{(1)}\tau^{(2)})(f_1f_2+g_1g_2)[(\boldsymbol{\sigma}^{(1)}\mathbf{n})(\boldsymbol{\sigma}^{(2)}\mathbf{n})-\tfrac{1}{3}(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})]\left(\frac{3}{r^3}+\frac{3\mu}{r^2}+\frac{\mu^2}{r}\right)e^{-\mu r}\cdot\varphi$$
$$+\frac{1}{4\pi M\mu}(\tau^{(1)}\tau^{(2)})(f_1f_2+g_1g_2)\cdot\tfrac{1}{3}(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})\frac{\mu^2}{r}e^{-\mu r}\cdot\varphi+\frac{12}{4\pi M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2\left(\frac{1}{r^3}+\frac{\mu}{r^2}\right)e^{-\mu r}\cdot B\varphi_{l=2}$$
$$-\frac{1}{4\pi M\mu}(\tau^{(1)}\tau^{(2)})g_1g_2[1+(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})]\mu^2\frac{e^{-\mu r}}{r}\cdot\varphi\quad(\mathbf{n}=\mathbf{r}/r).\quad(19)$$

Substituting (15), (17) into (11) and (19), using the relations $(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})=1$, and $(\tau^{(1)}\tau^{(2)})=-3$ for the ground state of the deuteron, and equating the coefficients of $\mathcal{Y}_0{}^m$ and of $\mathcal{Y}_2{}^m$, respectively, in the result, we obtain finally

$$-\frac{1}{M}\frac{d^2u}{dr^2}-\left[E_0+3\frac{g_1{}^2+g_2{}^2}{4\pi r}e^{-\mu r}\right]u-2\sqrt{2}\cdot\frac{1}{4\pi M\mu}(f_1f_2+g_1g_2)\left(\frac{3}{r^3}+\frac{3\mu}{r^2}+\frac{\mu^2}{r}\right)e^{-\mu r}v$$
$$-\frac{\mu}{4\pi M}(f_1f_2+g_1g_2)\frac{e^{-\mu r}}{r}u+\frac{6\mu}{4\pi M}g_1g_2\frac{e^{-\mu r}}{r}u=0,\quad(20)$$

$$-\frac{1}{M}\left(\frac{d^2v}{dr^2}-\frac{6v}{r^2}\right)v-\left[E_0+3\frac{g_1{}^2+g_2{}^2}{4\pi r}e^{-\mu r}\right]v-2\sqrt{2}\frac{1}{4\pi M\mu}(f_1f_2+g_1g_2)\left(\frac{3}{r^3}+\frac{3\mu}{r^2}+\frac{\mu^2}{r}\right)e^{-\mu r}u$$
$$+2\frac{1}{4\pi M\mu}(f_1f_2+g_1g_2)\left(\frac{3}{r^3}+\frac{3\mu}{r^2}+\frac{\mu^2}{r}\right)e^{-\mu r}v-\frac{\mu}{4\pi M}(f_1f_2+g_1g_2)\frac{e^{-\mu r}}{r}v$$
$$-\frac{9}{\pi M\mu}g_1g_2\left(\frac{1}{r^3}+\frac{\mu}{r^2}\right)e^{-\mu r}v+\frac{6\mu}{4\pi M}g_1g_2\frac{e^{-\mu r}}{r}v=0,\quad(21)$$

where $u=AR_0(r)$, $v=BR_2(r)$ so that

$$\int_0^\infty(u^2+v^2)dr=1.$$

(20) and (21) are the two radial equations for u and v derived from (11). It is seen that the inadmissible singularities appear with coefficients proportional to either $(g_1g_2+f_1f_2)$ or g_1g_2. To remove these singularities the only way is to put $g_1g_2=f_1f_2=0$. This is, however, equivalent to putting $f_1=g_1=0$, since f_2 and g_2 have already been fixed in Møller and Rosenfeld's original non-relativistic theory.

With the reappearance of the inadmissible singularity in the relativistic approximation, our wave equation (11) will give no stationary states in the rigorous solution unless some assumption equivalent to a cut-off is introduced. This difficulty is the same as that encountered by the single type of meson theories before the introduction of the Møller-Rosenfeld mixture. A cut-off is thus inevitable, and the application of the mixed theory is no longer justified as it gives no better service than the other simple theories. On the other hand, one can remove the inadmissible singularities by putting $f_1=g_1=0$, but then the tensor force disappears entirely from (20) and (21) and the observed quadrupole moment still remains unexplained as in the non-relativistic theory. It seems that the situation will remain the same in all the higher non-relativistic approximations, namely, whenever we obtain a tensor force it is always inherent with the inadmissible singularities. Furthermore, the removal of these singularities will then be impossible since all the interaction constants have already been fixed.

Without considering seriously the difficulties mentioned above, Møller and Rosenfeld, and later Møller himself,[10] tried to solve their equations corresponding to (11) which also contain the inadmissible singularities, by the perturbation method, considering the relativistic terms as the perturbed terms. They obtained convergent and unique results for the quadrupole moment in the first approximation. Their results thus obtained can only be considered as provisional. Some troubles are therefore bound to be encountered in the higher approximations in such a way that the perturbation cannot converge. To avoid this divergence, we have again to introduce the cut-off procedure as is inevitable in the rigorous solution. However, it seems quite likely that the arbitrary radius of cut-off will only influence sensitively the higher approximations where the results diverge originally before the introduction of the cut-off. This would mean that the results obtained by Møller, Rosenfeld, and Hulthén still give the principal part of the quadrupole moment, which is independent of the arbitrary radius of cut-off. Further discussion of this point will be the topic of another communication.

For the sake of completeness, the same perturbation method as used by Møller and Rosenfeld has also been applied to solve (11). We find

$$(E, l=2\,|\,P\,|\,E_0, l=0) \equiv \int \varphi_{l=2}^{E\dagger} P \varphi_{l=0}^{E_0} d^3x$$

$$= -\frac{2\sqrt{2}}{4\pi M\mu}(f_1 f_2 + g_1 g_2)\int_0^\infty \left(\frac{3}{r^3}+\frac{3\mu}{r^2}+\frac{\mu^2}{r}\right) e^{-\mu r} R_0^{E_0}(r) R_2^{E}(r) dr, \quad (22)$$

where E_0 is the deuteron binding energy and E is the continuous positive energy. The quadrupole moment is given by (cf. Møller and Rosenfeld's paper)

$$Q = \frac{\sqrt{2}}{5}\int_0^\infty \frac{(l=0, E_0\,|\,r^2\,|\,E, l=2)(E, l=2\,|\,P\,|\,E_0, l=0)+\text{conj.}}{E_0 - E} dE. \quad (23)$$

Substituting (22) into (23), we find that the result is identical with final expression (123) for quadrupole moment in Møller and Rosenfeld's paper. Therefore the quadrupole moment obtained from the present theory by the perturbation method is the same as that evaluated by Hulthén.

In conclusion, the author wishes to express his sincere thanks to Professor W. Pauli for suggesting the present problem and for his invaluable help during the investigation.

[10] C. Møller, Kgl. Danske Vid. Sels. Math.-Fys. Medd **18**, 6 (1941).

The Azimuthal Variations of Cosmic Radiation at 22° Latitude

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Four triple-coincidence cosmic-ray telescopes directed at a common zenith angle of 60° were mounted on a turntable. Each telescope represented a vertical plane 90° from that of its neighbor. In each was inserted 10.2 cm of lead. By 180° reversals each telescope was interchanged in its position with the one directly opposite. Countings were made at settings of the table for every 10 degree of azimuth angle. Azimuthal variations in the North-West and North-East quadrants check qualitatively those theoretically predicted by Hutner. The data are still not sufficient to warrant quantitative comparison. It has been found that the East-West asymmetry does not coincide with magnetic east-west plane, but occurs at 280°–100° plane.

DR. HUTNER[1] made a quantitative study of the azimuthal effect of cosmic radiation (variation of cosmic-ray intensity at a given point on the earth's surface and fixed zenith angle but for different azimuths), using a constant zenith angle of 60 degrees, at a geomagnetic latitude of 20° north. This geomagnetic latitude was chosen because the penumbra[2] (the reader is referred to Shremp's paper for reference for details about the penumbra) which makes possible the determination of the energy spectrum of the primary cosmic rays makes a significant contribution at intermediate latitudes (between 10° and 40° geomagnetic latitude). The author undertook to perform an experiment at Lahore (geomagnetic latitude 22° N) to find the predicted azimuthal effect for a fixed zenith angle of 60°. A preliminary report[3] on the experiment appeared in the *Physical Review* in 1941. This is therefore a progress report on the experiment.

In the present experiment four cosmic-ray telescopes were used, each consisting of three triple coincidence G-M counter tubes with lead filter of 10.2 cm. The distance between the centers of the first and the third tubes was 36 cm in each case. The solid angle range covered by each telescope was 8.07° in the vertical and 38.87° in the lateral planes. In the previous experiment[3] the solid angle range covered by each telescope was 14.5° in the vertical and 65° in the lateral planes. The telescopes were mounted on the four edges of a square table, which was rotated back and forth through an angle of 180° by hand at frequent intervals. The frequent interchanging of the opposite telescopes made it possible to compare the cosmic-ray intensities from opposite directions. The azimuths were changed by steps of 10° each time. Each of the four telescopes swept the whole sky for the fixed zenith angle at intervals of 10° at least twice, and some azimuths were covered more than twice. The two or more readings of one telescope at the same azimuth, though taken about a month and a half apart, do not differ from each other by more than their statistical fluctuations and thus showed the systematic working of the telescope for the whole experimental period. These readings were taken between October 31, 1941, and February 21, 1942.

Compared with the solid angle range of the earlier experiment, that of the new experiment is reduced giving therefore a resolving power three times as great. The number of counts per unit time for each new telescope should therefore be one third of the number of counts per unit time in the earlier experiment. The only other difference between the new telescopes and the old was that, among the four, some of the individual G-M tubes had been interchanged. Another rough check on the systematic working of the tubes was provided by the fact that the ratio of the counts per day in the new experiment to that in the old was 1:3.12.

The apparatus was installed in a special laboratory constructed for the purpose on top of the new physics laboratory of Forman Christian College. The roof of the observatory was made

[1] R. A. Hutner, Phys. Rev. **55**, 614 (1939).
[2] E. J. Shremp, Phys. Rev. **54**, 157 (1938).
[3] P. S. Gill, Phys. Rev. **60**, 153 (1941).

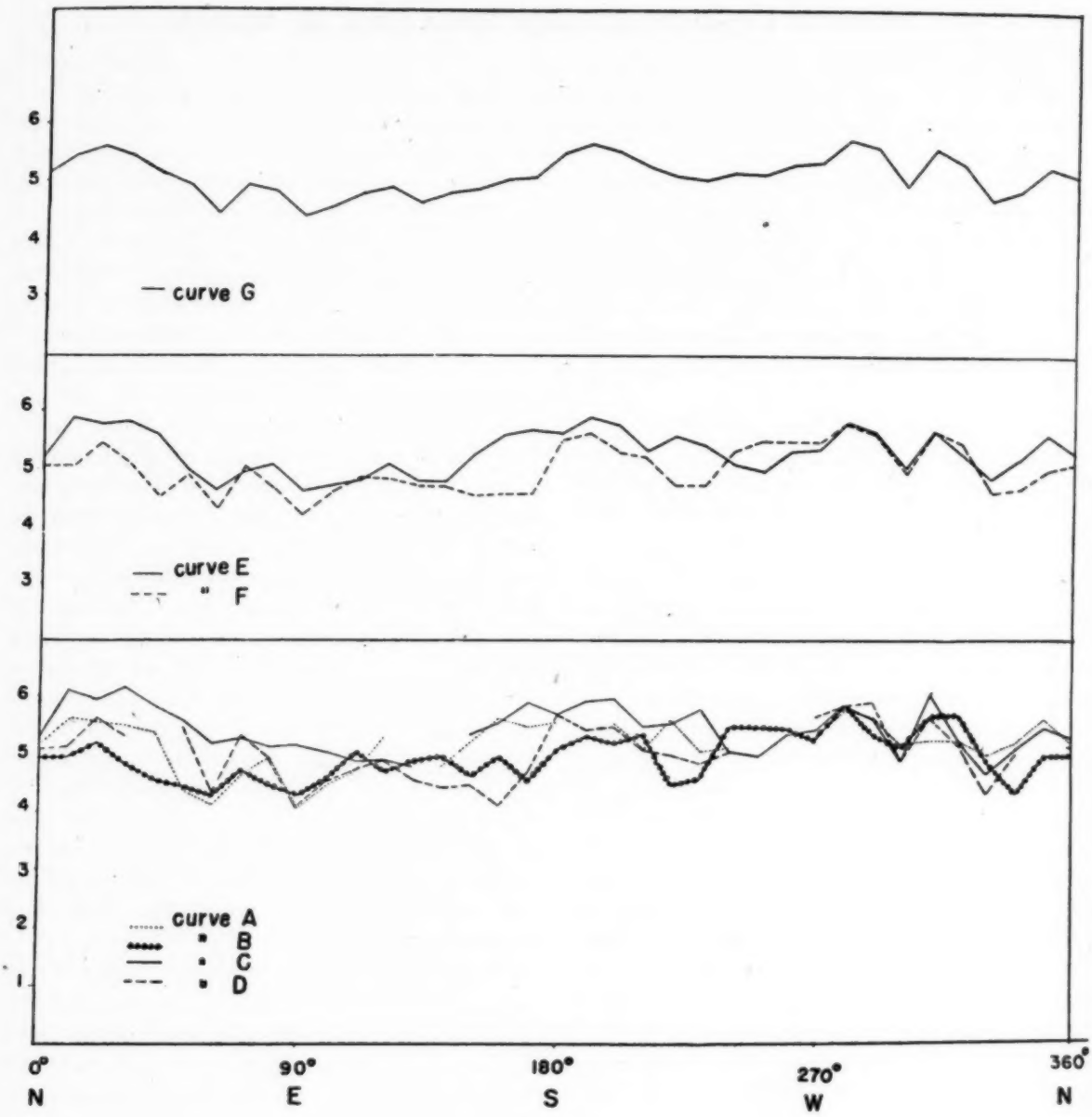

FIG. 1.

of a single sheet of galvanized iron and was thus virtually open to the sky.

INTERPRETATION OF DATA

Curves *A*, *B*, *C*, and *D* of Fig. 1 represent the number of triple coincidences per hour obtained by telescopes *A*, *B*, *C*, and *D* for each 10° interval of azimuths for the fixed zenith angle of 60°. Each telescope made a complete turn of the horizon, except for the azimuths where one of the amplifying tubes or one of the power tubes or one of the G-M tubes was not functioning properly. These exceptions are indicated by the gaps in the graphs. An examination of these four curves clearly shows the marked irregularities in the north-west quadrant. The humps in the curves in this quadrant occur at the same angles in each curve. No such marked variations at consistent azimuth angles are shown in other quadrants.

Curves *A* and *C* represent the data collected by opposite telescopes *A* and *C* during the same period, and curve *E* is the mean of *A* and *C*.

Similarly curve F is the mean of B and D and represents the data collected by two opposite telescopes B and D during the same period. In other words each point of curve F represents data taken about forty days after data taken for the corresponding point of curve E. In taking the means, it is assumed that the efficiency of the two opposite telescopes in each case is the same. By that the number of counts at each angle is doubled, and therefore the statistical errors are reduced. It should be noted that in the N-W quadrant all the points of curve F coincide with those of curve E within the probable errors except at 340°–350°. Even for these angles the general form of the curves is the same. The parallelism between curves E and F is clear enough, with the exception of regions between 140°–170° and between 210°–250°.

Curve G is the weighted mean of all the four telescopes. The means of all the ordinates of curve A, and similarly of curves B, C, and D, were taken separately. These means were weighted with reference to the mean of all the ordinates of the four curves (A, B, C, D) and it turned out to be the same as that of telescope A. That is, all the points of curves B, C, and D were multiplied by the ratio of their respective means to the general mean or that of curve A. It is again to be noted that in the N-W quadrant curve G has marked azimuthal irregularities. Two humps, one at 280° and the other at 310°, are prominent in this quadrant. The curve in the N-E quadrant varies considerably less. This is in agreement with Hutner's conclusion that in the northern hemisphere, the curve between the N- and E-directions should be less irregular because of the supposed predominance of positive ρ articles.

TABLE I. Counts per minute in N-W quadrant (N_A) and in the S-E quadrant (N_B).

Angle	N_A	Angle	N_B	$\alpha=2\left(\frac{N_A-N_B}{N_A+N_B}\right)$
0	0.272±0.005	180	0.298±0.006	−0.091±0.027
350	0.263±0.005	170	0.280±0.005	−0.062±0.026
340	0.285±0.005	160	0.273±0.005	+0.043±0.028
330	0.262±0.005	150	0.267±0.005	−0.019±0.027
320	0.274±0.005	140	0.267±0.005	+0.026±0.026
310	0.280±0.004	130	0.290±0.004	−0.035±0.016
300	0.301±0.004	120	0.278±0.004	+0.080±0.020
290	0.301±0.004	110	0.272±0.004	+0.101±0.020
280	0.288±0.004	100	0.252±0.004	+0.133±0.016
270	0.276±0.004	90	0.256±0.004	+0.075±0.016

Hutner's results refer strictly to primaries before they enter the atmosphere. In the present experiment the secondaries were filtered out by 10.2 cm of lead, and therefore the telescopes recorded the passage of hard secondaries presumably having the same direction as the primaries which were responsible for producing them.

THE EAST-WEST AND NORTH-SOUTH ASYMMETRIES

Special cases of the "Azimuthal effect" are the E-W and N-S asymmetries. Tables I, II, and III give the experimental results on these asymmetries at Lahore (geomagnetic latitude 22° north) for a fixed zenith angle of 60°. Table I is the same as Table I already published,[3] except for the probable errors, which were not correctly calculated. Table II gives the results of the new experiment. (x) represents the counts per hour for each angle from N-E-S, while (y) gives the counts per hour from S-W-N.

$$\text{The asymmetry} = 2(x-y)/(x+y),$$

is calculated at each 10° interval.

A notable feature of these asymmetries is that the maximum E-W difference does not coincide with the magnetic E-W plane, but is found to occur in the 280°–100° plane. Each of the four telescopes gives the maximum E-W difference at the same angle (280°–100°). This becomes more significant when we look at Table I, which also

TABLE II. Counts per hour from N-E-S(x) and from S-W-N(y).

Angle	x	Angle	y	$\alpha=2\left(\frac{x-y}{x+y}\right)$
0	5.14±0.10	180	5.53±0.12	−0.073±0.029
10	5.46±0.11	190	5.69±0.12	−0.041±0.029
20	5.60±0.13	200	5.52±0.12	+0.014±0.032
30	5.44±0.13	210	5.26±0.11	+0.021±0.032
40	5.19±0.15	220	5.12±0.13	+0.013±0.038
50	4.96±0.13	230	5.04±0.13	−0.016±0.037
60	4.47±0.12	240	5.18±0.12	−0.147±0.035
70	4.98±0.13	250	5.16±0.16	−0.035±0.040
80	4.86±0.13	260	5.30±0.20	−0.087±0.046
90	4.44±0.09	270	5.40±0.12	−0.195±0.030
100	4.62±0.09	280	5.80±0.09	−0.226±0.025
110	4.83±0.11	290	5.61±0.11	−0.149±0.030
120	4.92±0.12	300	4.99±0.10	−0.014±0.031
130	4.73±0.12	310	5.60±0.14	−0.167±0.035
140	4.81±0.18	320	5.32±0.14	−0.100±0.045
150	4.89±0.11	330	4.70±0.11	+0.040±0.032
160	5.05±0.13	340	4.88±0.10	+0.034±0.033
170	5.10±0.12	350	5.29±0.16	−0.037±0.038

TABLE III.

Angle	A	B	C	D	E	F	G
0	5.18 = 0.18	4.95 = 0.16	5.28 = 0.22	5.15 = 0.22	5.23 = 0.14	5.05 = 0.14	5.14 = 0.10
10	5.60 = 0.25	4.97 = 0.18	6.12 = 0.30	5.17 = 0.22	5.86 = 0.20	5.07 = 0.14	5.46 = 0.11
20	5.55 = 0.23	5.22 = 0.18	5.98 = 0.24	5.63 = 0.33	5.76 = 0.17	5.42 = 0.19	5.60 = 0.13
30	5.48 = 0.23	4.84 = 0.21	6.17 = 0.30	5.30 = 0.32	5.82 = 0.19	5.07 = 0.19	5.44 = 0.13
40	5.37 = 0.22	4.53 = 0.29	5.83 = 0.25		5.60 = 0.17	4.53 = 0.29	5.19 = 0.15
50	4.38 = 0.20	4.46 = 0.29	5.57 = 0.23	5.42 = 0.32	4.98 = 0.15	4.94 = 0.22	4.96 = 0.13
60	4.11 = 0.18	4.25 = 0.28	5.18 = 0.22	4.35 = 0.29	4.64 = 0.14	4.30 = 0.20	4.47 = 0.12
70	4.58 = 0.30	4.74 = 0.21	5.29 = 0.22	5.30 = 0.32	4.94 = 0.18	5.02 = 0.19	4.98 = 0.13
80	4.94 = 0.22	4.48 = 0.17	5.13 = 0.31	4.88 = 0.30	5.04 = 0.19	4.68 = 0.18	4.86 = 0.13
90	4.17 = 0.15	4.28 = 0.20	5.15 = 0.18	4.17 = 0.20	4.66 = 0.12	4.22 = 0.14	4.44 = 0.09
100	4.40 = 0.14	4.59 = 0.16	5.00 = 0.16	4.50 = 0.22	4.70 = 0.11	4.54 = 0.14	4.62 = 0.09
110	4.68 = 0.17	5.02 = 0.22	4.88 = 0.21	4.70 = 0.21	4.78 = 0.14	4.86 = 0.15	4.83 = 0.11
120	5.28 = 0.32	4.71 = 0.21	4.86 = 0.18	4.83 = 0.21	5.07 = 0.18	4.77 = 0.15	4.92 = 0.12
130		4.87 = 0.21	4.77 = 0.22	4.54 = 0.21	4.77 = 0.22	4.70 = 0.15	4.73 = 0.12
140	4.75 = 0.30	4.97 = 0.31		4.42 = 0.29	4.75 = 0.30	4.70 = 0.21	4.81 = 0.18
150	5.17 = 0.22	4.62 = 0.21	5.32 = 0.22	4.47 = 0.18	5.24 = 0.22	4.54 = 0.14	4.89 = 0.11
160	5.59 = 0.19	4.97 = 0.31	5.52 = 0.23	4.13 = 0.29	5.56 = 0.15	4.55 = 0.21	5.05 = 0.13
170	5.45 = 0.19	4.50 = 0.29	5.84 = 0.24	4.67 = 0.21	5.64 = 0.15	4.58 = 0.18	5.10 = 0.12
180	5.52 = 0.18	5.16 = 0.19	5.71 = 0.23	5.71 = 0.33	5.62 = 0.15	5.44 = 0.19	5.53 = 0.12
190		5.27 = 0.14	5.88 = 0.22	5.90 = 0.24	5.88 = 0.22	5.58 = 0.14	5.69 = 0.12
200	5.52 = 0.23	5.17 = 0.18	5.93 = 0.25	5.46 = 0.32	5.72 = 0.17	5.32 = 0.18	5.52 = 0.12
210	5.19 = 0.22	5.32 = 0.18	5.46 = 0.23	5.07 = 0.22	5.32 = 0.16	5.20 = 0.14	5.26 = 0.11
220	5.58 = 0.23	4.44 = 0.27	5.50 = 0.23	4.97 = 0.31	5.54 = 0.16	4.70 = 0.21	5.12 = 0.13
230	5.04 = 0.22	4.55 = 0.29	5.75 = 0.23	4.84 = 0.30	5.40 = 0.16	4.70 = 0.21	5.04 = 0.13
240	5.10 = 0.22	5.45 = 0.23	5.02 = 0.19	5.10 = 0.31	5.06 = 0.15	5.28 = 0.19	5.18 = 0.12
250		5.42 = 0.23	4.96 = 0.22		4.96 = 0.22	5.42 = 0.23	5.16 = 0.16
260		5.42 = 0.32	5.27 = 0.24		5.27 = 0.24	5.42 = 0.32	5.30 = 0.20
270	5.26 = 0.19	5.26 = 0.23	5.40 = 0.17	5.66 = 0.33	5.33 = 0.13	5.46 = 0.20	5.40 = 0.12
280	5.73 = 0.15	5.79 = 0.21	5.85 = 0.19	5.80 = 0.18	5.79 = 0.12	5.80 = 0.14	5.80 = 0.09
290	5.65 = 0.23	5.30 = 0.21	5.57 = 0.19	5.88 = 0.24	5.61 = 0.15	5.59 = 0.16	5.61 = 0.11
300	5.13 = 0.22	5.13 = 0.22	4.88 = 0.18	4.79 = 0.21	5.00 = 0.14	4.96 = 0.15	4.99 = 0.10
310	5.23 = 0.32	5.65 = 0.23	6.00 = 0.34	5.54 = 0.23	5.62 = 0.23	5.60 = 0.16	5.60 = 0.14
320	5.21 = 0.32	5.68 = 0.23	5.21 = 0.32	5.13 = 0.22	5.21 = 0.22	5.40 = 0.16	5.32 = 0.14
330	4.98 = 0.22	4.83 = 0.19	4.63 = 0.30	4.32 = 0.20	4.80 = 0.18	4.58 = 0.14	4.70 = 0.11
340	5.15 = 0.22	4.30 = 0.20	5.10 = 0.18	4.98 = 0.22	5.12 = 0.14	4.64 = 0.15	4.88 = 0.10
350	5.58 = 0.23	4.93 = 0.30	5.46 = 0.32		5.52 = 0.20	4.93 = 0.30	5.29 = 0.16

shows that the maximum E-W difference occurs in 280°–100° plane. This point is being further investigated in this laboratory and it is hoped that the results will be reported in the near future.

The present survey will be continued with a much larger apparatus, which is being constructed for rapidly collecting new data. The author wishes to express his thanks to the University of the Punjab for granting a fellowship for continuing these experiments at Lahore, to the University of Chicago for supplying the apparatus, and to Forman Christian College for the facilities of the Physics Department.

Demagnetizing Factors of the General Ellipsoid

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Charts and tables of the demagnetizing factors of prolate and oblate spheroids are readily available; however, demagnetizing factors of ellipsoids of three different axes are incompletely tabulated and laborious to calculate. This article presents charts and tables which make possible easy determination of the demagnetizing factor for any principal axis of an ellipsoid of any shape. Formulas for the demagnetizing factors of the general ellipsoid are included together with supplementary formulas which cover a large number of special cases.

I. INTRODUCTION

IF a ferromagnetic body of irregular shape is brought into a uniform applied field H_0, the magnetizing force H inside the material differs in magnitude from the applied field and varies in direction throughout the body in an unknown manner. This is usually a great disadvantage in research on magnetic materials, because such research usually requires studies of the relation between the magnetization J and the magnetizing force. However, for homogeneous bodies whose surface is of the second degree,[1] H and J (after suitable magnetic treatment) are uniform throughout, though they are not necessarily the same in direction as H_0. The ellipsoid has the only surface of the second degree that is finite, so materials in this form are frequently used in precise investigations of magnetic materials.

Inside any ellipsoid the component of H along any principal axis i is determined by the relation

$$H_i=(H_0)_i-N_iJ_i, \quad i=x, y, z, \qquad (1.1)$$

where N_i is a constant called the demagnetizing factor; it is determined by the ratios of the axes. Hence, if the magnetization components and the lengths of the axes of the ellipsoid are known, H inside the body may be found.[2]

The determination of N is rather involved, except for the special cases of ellipsoids of revolution. Therefore, it is believed that calculation of N for various axial ratios of the *general* ellipsoid will serve a useful purpose. Naturally, it is impossible to include values of the demagnetizing factor for all axial ratios, so graphs of N have been made that will allow interpolation for those ratios not explicitly included. The following section gives the formulas from which the demagnetizing factors have been calculated, together with comments on the method by which they were obtained.

II. FORMULAS FOR THE DEMAGNETIZING FACTORS OF THE GENERAL ELLIPSOID

The equations given below are for the demagnetizing factors along the three axes of the general ellipsoid under the assumption that

$$a\geq b\geq c\geq 0,$$

where a, b, and c are the ellipsoid semi-axes. The demagnetizing factors (*corresponding to the semi-axes a, b, and c*) will be labeled L, M, and N. Then, as indicated in Eq. (1.1), to find the component of the magnetizing force along any principal axis, the component of the applied field $(H_0)_i$ along that axis must be considered together with the appropriate magnetization component J_i and demagnetizing factor. The formulas for L, M, and N are

$$L/4\pi=\frac{\cos\varphi\cos\vartheta}{\sin^3\vartheta\sin^2\alpha}[F(k,\vartheta)-E(k,\vartheta)], \qquad (2.1)$$

$$M/4\pi=\frac{\cos\varphi\cos\vartheta}{\sin^3\vartheta\sin^2\alpha\cos^2\alpha}\Big[E(k,\vartheta)-\cos^2\alpha F(k,\vartheta)-\frac{\sin^2\alpha\sin\vartheta\cos\vartheta}{\cos\varphi}\Big], \qquad (2.2)$$

[1] J. C. Maxwell, *Electricity and Magnetism* (The Clarendon Press, Oxford, 1904), third edition, Vol. 2, pp. 66–70.

[2] It is possible to find H at the center of a cylindrical bar if a kind of demagnetizing factor different from that for the ellipsoid is defined. The use of these so called *ballistic* demagnetizing factors is discussed by R. M. Bozorth and D. M. Chapin, J. App. Phys. **12**, 320 (1942).

$$N/4\pi=\frac{\cos\varphi\cos\vartheta}{\sin^3\vartheta\cos^2\alpha}\left[\frac{\sin\vartheta\cos\varphi}{\cos\vartheta}-E(k,\vartheta)\right], \quad (2.3)$$

where

$$\cos\vartheta=c/a, \quad (0\leq\vartheta\leq\pi/2), \quad (2.4)$$

$$\cos\varphi=b/a, \quad (0\leq\varphi\leq\pi/2), \quad (2.5)$$

$$\sin\alpha=\left[\frac{1-(b/a)^2}{1-(c/a)^2}\right]^{\frac{1}{2}}=\frac{\sin\varphi}{\sin\vartheta}=k, \quad (0\leq\alpha\leq\pi/2), \quad (2.6)$$

and $F(k,\vartheta)$ and $E(k,\vartheta)$ are elliptic integrals of the first and second kinds:[3] k is the modulus, and ϑ is the amplitude of these integrals. Equations (2.1)–(2.3) are given in forms that are convenient for calculation. Methods of obtaining general expressions for L, M, and N of the general ellipsoid are available in several places.[1,4] Maxwell has shown how the method of Poisson[5] for getting the potential of a uniformly magnetized body yields the demagnetizing factors for the general ellipsoid. The equations for the factors are not given in directly usable form however. Kellogg[4] briefly indicates the method of obtaining L, M, and N and gives expressions similar to Eqs. (2.1)–(2.3). The most complete discussion of the solution for the demagnetizing factors of the general ellipsoid by use of the Poisson method is given by Poritsky.[4] Stratton[4] obtains expressions for L, M, and N in the same form as found in Maxwell.

A useful relationship that exists between the demagnetizing factors of an ellipsoid is

$$L+M+N=4\pi. \quad (2.7)$$

This may be proved by using the following expression for the gravitational potential of an ellipsoid of uniform density ρ:

$$V=(1/2)\rho(c-Lx^2-My^2-Nz^2), \quad (2.8)$$

where the coefficients L, M, and N represent the usual demagnetizing factors as in the Poisson method. Poisson's operation

$$\nabla^2V=\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\frac{\partial^2V}{\partial z^2}=-4\pi\rho, \quad (2.9)$$

gives Eq. (2.7).

If the ellipsoid is an ellipsoid of rotation, i.e., a prolate or oblate spheroid, Eqs. (2.1)–(2.3) reduce to more simple forms; the formulas for these and other limiting cases are given below:

(*a*) If $b=c$ (prolate spheroid),

$$L/4\pi=\frac{1}{m^2-1}\left[\frac{m}{2(m^2-1)^{\frac{1}{2}}}\times\ln\left(\frac{m+(m^2-1)^{\frac{1}{2}}}{m-(m^2-1)^{\frac{1}{2}}}\right)-1\right], \quad (2.10)$$

where $m=a/c$.

$$M/4\pi=N/4\pi=\frac{m}{2(m^2-1)}\left[m-\frac{1}{2(m^2-1)^{\frac{1}{2}}}\times\ln\left(\frac{m+(m^2-1)^{\frac{1}{2}}}{m-(m^2-1)^{\frac{1}{2}}}\right)\right]. \quad (2.11)$$

If $m\gg1$ (very slender prolate spheroid),

$$L/4\pi\simeq(1/m^2)(\ln 2m-1), \quad (2.12)$$

$$M/4\pi=N/4\pi\simeq\tfrac{1}{2}[1-(\ln 2m-1)/m^2]. \quad (2.13)$$

(*b*) If $a\gg b\geq c$ (very slender ellipsoid),

$$L/4\pi=(bc/a^2)(\ln 4a/(b+c)-1), \quad (2.14)$$

$$M/4\pi=c/(b+c)-\tfrac{1}{2}(bc/a^2)\ln 4a/(b+c)+bc(3b+c)/4a^2(b+c), \quad (2.15)$$

$$N/4\pi=b/(b+c)-\tfrac{1}{2}(bc/a^2)\ln[4a/(b+c)]+bc(b+3c)/4a^2(b+c). \quad (2.16)$$

(*c*) If $a=\infty$, $b\geq c$ (elliptic cylinder),

$$M/4\pi=c/(b+c), \quad (2.17)$$

$$N/4\pi=b/(b+c). \quad (2.18)$$

(*d*) If $a=b$ (oblate spheroid),

$$L/4\pi=M/4\pi=\frac{1}{2(m^2-1)}\{m^2(m^2-1)^{-\frac{1}{2}}\times\arcsin[(m^2-1)^{\frac{1}{2}}/m]-1\}, \quad (2.19)$$

[3] See B. O. Pierce, *A Short Table of Integrals*, third edition (Ginn and Company, New York, 1929), p. 66 *et seq.*

[4] O. Kellogg, *Foundations of Potential Theory* (Verlagsbuchhandlung Julius Springer, Berlin, 1929), pp. 192–194, also p. 197, Exercise 7. H. Poritsky, *Magnetization of an Ellipsoid* (General Electric Company, Schenectady, New York). J. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 211–213.

[5] Poisson showed that if V is the gravitational potential of a body of uniform density ρ, the magnetic potential Ω_x of the same body is $-\partial V/\partial x$, if it is uniformly magnetized in the x-direction so that $J=\rho$.

TABLE I. Demagnetizing factors of the general ellipsoid calculated from formulas of section II.

c/a	b/a	L/4π	M/4π	N/4π
.017452	.99620		.013471	
	.98164		.013719	
	.95632		.014165	
	.91357		.014989	
	.40709		.037634	
	.29285		.053078	
	.19158		.080810	
	.071895		.19374	
	.055162		.23901	
	.039018		.30809	
.034900	.89892			.94467
	.79891			.94104
	.69511			.93617
	.60246			.93029
	.50091			.92126
	.40798		.071847	.90905
	.29427		.099697	.88398
	.19386		.14720	.83986
	.093838		.26746	.72453
	.077961		.30596	
	.062880		.35411	
.087156	1.00000	.06154	.06154	.87692
	.98492	.06108	.06281	.87611
	.94013	.06051	.06620	.87329
	.86727	.05908	.07222	.86870
	.76809	.05683	.08307	.86010
	.64625	.05347	.10028	.84626
	.50565	.04846	.12912	.82242
	.35171	.04087	.18185	.77728
	.19376	.02898	.29661	.67441
	.08716	.01641	.49180	.49180
.17365	1.00000	.11138	.11138	.77724
	.98527	.11085	.11327	.77588
	.94157	.10936	.11903	.77161
	.87037	.10665	.12955	.76380
	.77413	.10245	.14651	.75104
	.65641	.09627	.17296	.73077
	.52213	.08734	.21496	.69770
	.37895	.07455	.28191	.64354
	.24371	.05756	.38888	.55356
	.17365	.04582	.47709	.47709
.25882	1.00000	.15207	.15207	.69587
	.98583	.15136	.15440	.69424
	.94386	.14927	.16154	.68919
	.87562	.14553	.17452	.67995
	.78392	.13985	.19464	.66551
	.67269	.13167	.22498	.64335
	.54793	.12029	.26973	.60998
	.41970	.10524	.33356	.56120
	.30844	.08817	.41326	.49857
	.25882	.07892	.46054	.46054
.34202	1.00000	.18555	.18555	.62889
	.98660	.18477	.18805	.62718
	.94695	.18221	.19597	.62181
	.88274	.17755	.21045	.61200
	.79697	.17108	.23128	.59764
	.69414	.16176	.26200	.57624
	.58115	.14940	.30440	.54620
	.46933	.13428	.35885	.50687
	.37895	.11924	.41586	.46490
	.34202	.11218	.44391	.44391
.50000	1.00000	.23640	.23640	.52720
	.98863	.23555	.23885	.52560
	.95513	.23256	.24672	.52072
	.90139	.22760	.26012	.51228
	.83073	.22043	.27966	.49991
	.74825	.21105	.30560	.48335
	.66144	.19970	.33745	.46285
	.58115	.18772	.37190	.44037
	.52213	.17762	.40115	.42123
	.50000	.17356	.41322	.41322
.64279	1.00000	.27187	.27187	.45627
	.99111	.27097	.27403	.45499
	.96507	.26824	.28058	.45118
	.92374	.26370	.29152	.44478
	.87037	.25760	.30650	.43590
	.80971	.24999	.32522	.42479
	.74825	.24148	.34617	.41235
	.69414	.23330	.36647	.40023
	.65641	.22717	.38169	.39114
	.64279	.22486	.38757	.38757
.76604	1.00000	.29670	.29670	.40660
	.99375	.29598	.29830	.40572
	.97553	.29388	.30303	.40309
	.94696	.29048	.31062	.39890
	.91065	.28595	.32086	.39319
	.87037	.28064	.33285	.38651
	.83073	.27513	.34529	.37958
	.79697	.27018	.35644	.37338
	.77413	.26668	.36464	.36868
	.76604	.26542	.36729	.36729
.86603	1.00000	.31380	.31380	.37240
	.99622	.31352	.31462	.37186
	.98527	.31225	.31740	.37035
	.96824	.30983	.32224	.36789
	.94696	.30707	.32803	.36490
	.92374	.30396	.33473	.36131
	.90139	.30093	.34121	.35786
	.88274	.29825	.34692	.35484
	.87037	.29651	.35059	.35290
	.86603	.29584	.35208	.35208
.93969	1.00000	.32497	.32497	.35006
	.99823	.32474	.32548	.34978
	.99314	.32409	.32673	.34918
	.98527	.32306	.32889	.34805
	.97553	.32177	.33153	.34669
	.96507	.32038	.33435	.34527
	.95513	.31902	.33707	.34391
	.94695	.31790	.33947	.34260
	.94157	.31716	.34105	.34183
	.93969	.31690	.34155	.34155
.98481	1.00000	.33119	.33119	.33762
	.99955	.33120	.33143	.33737
	.99824	.33100	.33180	.33720
	.99622	.33078	.33230	.33692
	.99375	.33044	.33295	.33661
	.99111	.33010	.33364	.33628
	.98863	.32977	.33431	.33590
	.98660	.32950	.33487	.33563
	.98527	.32931	.33512	.33558
	.98481	.32925	.33537	.33537

TABLE II. Demagnetizing factors of the general ellipsoid for b/a equal to integral multiples of 0.1.

b/a	c/a	$L/4\pi$	$M/4\pi$	$N/4\pi$
.1000	.0872	.0181	.4624	.5194
	.0349		.2545	.7359
	.0175		.1530	
.2000	.1736	.0505	.4400	.5082
	.0872	.0294	.2890	.6828
	.0349		.1431	.8439
	.0175		.0785	
.3000	.2588	.0866	.4204	.4927
	.1736	.0653	.3369	.5990
	.0872	.0375	.2095	.7580
	.0349		.0977	.8854
	.0175		.0518	
.4000	.3420	.1230	.4014	.4760
	.2588	.1031	.3459	.5522
	.1736	.0768	.2692	.6532
	.0872	.0435	.1613	.7940
	.0349		.0727	.9083
	.0175		.0377	
.5000	.5000	.1735	.4132	.4132
	.3420	.1387	.3428	.5186
	.2588	.1149	.2920	.5930
	.1736	.0855	.2235	.6909
	.0872	.0481	.1308	.8217
	.0349			.9202
.6000	.5000	.1906	.3638	.4457
	.3420	.1516	.2962	.5521
	.2588	.1253	.2501	.6253
	.1736	.0926	.1895	.7173
	.0872	.0520	.1080	.8398
	.0349			.9301
.7000	.6428	.2341	.3641	.4018
	.5000	.2050	.3226	.4724
	.3420	.1625	.2600	.5776
	.2588	.1338	.2168	.6494
	.1736	.0987	.1622	.7385
	.0872	.0551	.0925	.8528
	.0349			.9364
.8000	.7660	.2706	.3555	.3739
	.6428	.2487	.3284	.4229
	.5000	.2172	.2887	.4941
	.3420	.1713	.2303	.5984
	.2588	.1408	.1908	.6684
	.1736	.1037	.1420	.7546
	.0872	.0576	.0798	.8630
	.0349			.9412
.9000	.8660	.3008	.3415	.3577
	.7660	.2845	.3240	.3915
	.6428	.2611	.2981	.4408
	.5000	.2275	.2605	.5121
	.3420	.1788	.2063	.6148
	.2588	.1470	.1697	.6832
	.1736	.1077	.1250	.7670
	.0872	.0597	.0691	.8710
	.0349			.9445
1.0000	1.0000	.3333	.3333	.3333
	.9848	.3314	.3312	.3374
	.9397	.3249	.3250	.3500
	.8660	.3138	.3138	.3724
	.7660	.2965	.2967	.4066
	.6428	.2718	.2718	.4561
	.5000	.2364	.2364	.5270
	.3420	.1856	.1858	.6288
	.2588	.1520	.1520	.6960
	.1736	.1114	.1117	.7775
	.0872	.0615	.0615	.8769

$$N/4\pi = m^2/(m^2-1)\{1-1/(m^2-1)^{\frac{1}{2}} \times \arcsin[(m^2-1)^{\frac{1}{2}}/m]\}. \quad (2.20)$$

If $m \gg 1$ (very flat oblate spheroid),

$$L/4\pi = M/4\pi \simeq (\pi/4m)(1-4/\pi m), \quad (2.21)$$

$$N/4\pi \simeq 1-\pi/2m+2/m^2. \quad (2.22)$$

(e) If $a \geq b \gg c$ (very flat ellipsoid),

$$L/4\pi = \frac{c}{a}\frac{1}{(1-e^2)^{\frac{1}{2}}}\frac{K-E}{e^2}, \quad (2.23)$$

$$M/4\pi = \frac{c}{a}\frac{E-(1-e^2)K}{e^2(1-e^2)^{\frac{1}{2}}}, \quad (2.24)$$

$$N/4\pi = 1-\frac{cE}{a(1-e^2)^{\frac{1}{2}}}, \quad (2.25)$$

where K and E are complete elliptic integrals[3] whose argument is

$$e = (1-b^2/a^2)^{\frac{1}{2}}. \quad (2.26)$$

III. CONSTRUCTION AND USE OF TABLES AND GRAPHS

Construction

The values of the demagnetizing factors for the general ellipsoid given in Table I are determined by the quantities ϑ and k ($=\sin\alpha$) used in Eqs. (2.1)–(2.3). These were chosen so that ϑ and α covered the region from 0° to 70° in 10° steps, and from 70° to 90°, ϑ varied in 5° and α in 10° steps. Some gaps in the values of the demagnetizing factors were revealed when the final graphs were plotted, as explained below; hence, additional values were subsequently computed to make the graphs complete. The axial ratios of the ellipsoids were then determined by Eqs. (2.4)–(2.6). $L/4\pi$ and $N/4\pi$ were determined directly from Eqs.

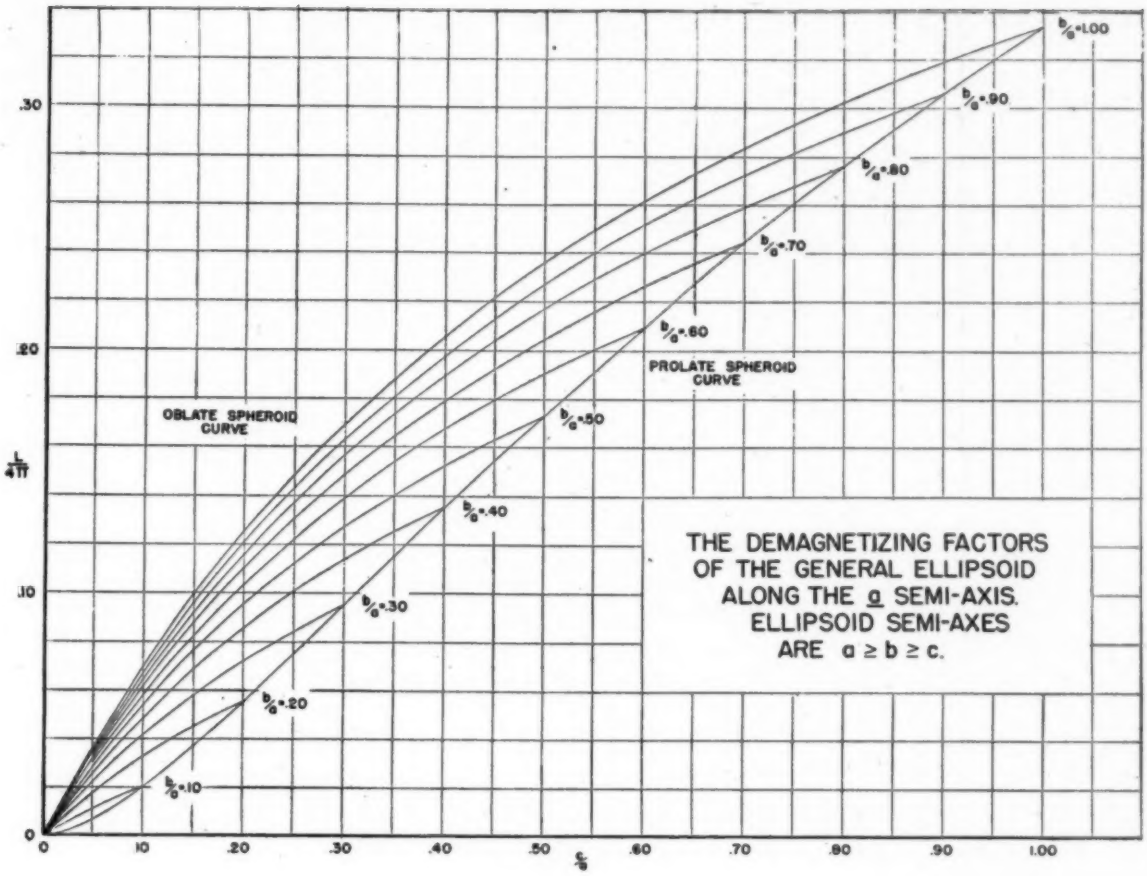

FIG. 1.

(2.1) and (2.3), and $M/4\pi$ was then determined in most cases by means of Eq. (2.7).

$L/4\pi$ and $N/4\pi$ in Table I are accurate to at least the fourth decimal place. $M/4\pi$ is somewhat less accurate than the other factors because it involves a difference.

The values of c/a and b/a obtained from ϑ and α are such that interpolation for the demagnetizing factors of other ellipsoids is not easy. To reduce this difficulty, each factor was separately plotted on a large scale against b/a, with c/a as a parameter. This gave a family of curves for each factor, L, M, and N. Each family of curves was then considered by itself. A value of b/a equal to an integral multiple of 0.1 was selected, and a value of the demagnetizing factor in question read from each curve. The constant value of b/a was now taken as a parameter and a new curve plotted with the factors (read from the above family of curves) as ordinates and the values of c/a (the parameter of the above family of curves) as abscissas. This process was repeated for different b/a values until a new family of curves was obtained with the parameter b/a equal to integral multiples of 0.1 from 0.1 to 1. These curves (shown in Figs. 1–3) warrant graphical interpolation to three decimal places. Table II contains the values of $L/4\pi$, $M/4\pi$, and $N/4\pi$ for various c/a (with b/a as a parameter) from which Figs. 1–3 were plotted. These values are accurate to three decimal places and are probably in error several units in the fourth place.

Use of Graphs

To use the graphs, any value of c/a may be chosen within the limitations

$$a \geq b \geq c \geq 0$$

and b/a may be found by interpolation. It is most important to remember that: *L corresponds*

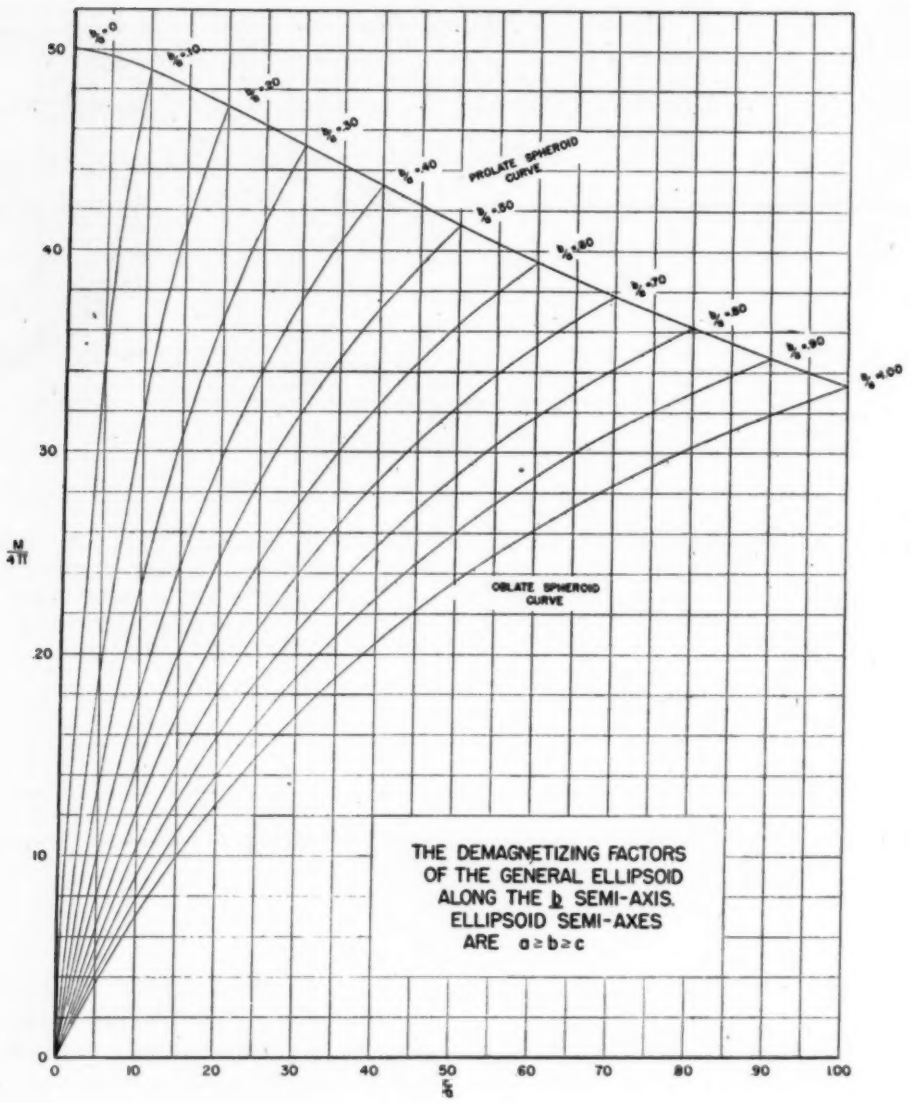

FIG. 2.

to a, the longest semi-axis; M corresponds to b, the intermediate semi-axis; and N corresponds to c, the shortest semi-axis. Two examples are given below illustrating the procedure for use of the graphs. Examples: (1) Suppose the semi-axes of an ellipsoid are 4, 3, and 2 and it is desired to find the demagnetizing factor along the axis of length 3. Then

$$a=4; \quad b=3; \quad \text{and} \quad c=2.$$

The factor is then read from the curves for $M/4\pi$ with axial ratios $c/a=0.500$ and $b/a=0.750$. $M/4\pi=0.306$ when read from the curve. (2) Suppose the axis lengths are 3, 2, and 1 and the factor along length 3 is wanted. Here

$$a=3; \quad b=2; \quad \text{and} \quad c=1.$$

In this case the factor is read from the curves for $L/4\pi$ with axial ratios $c/a=0.333$ and $b/a=0.667$. $L/4\pi=0.156$ when read from the curve.

IV. APPLICATION TO SPECIAL MATERIALS

If the magnetization curve is linear, and if the material either is isotropic or has its principal axes of magnetic anisotropy along the principal axes of the ellipsoid, then

$$J_i=\chi_i H_i=(\mu_i-1)H_i/4\pi, \qquad (4.1)$$

where the susceptibility χ_i and the permeability

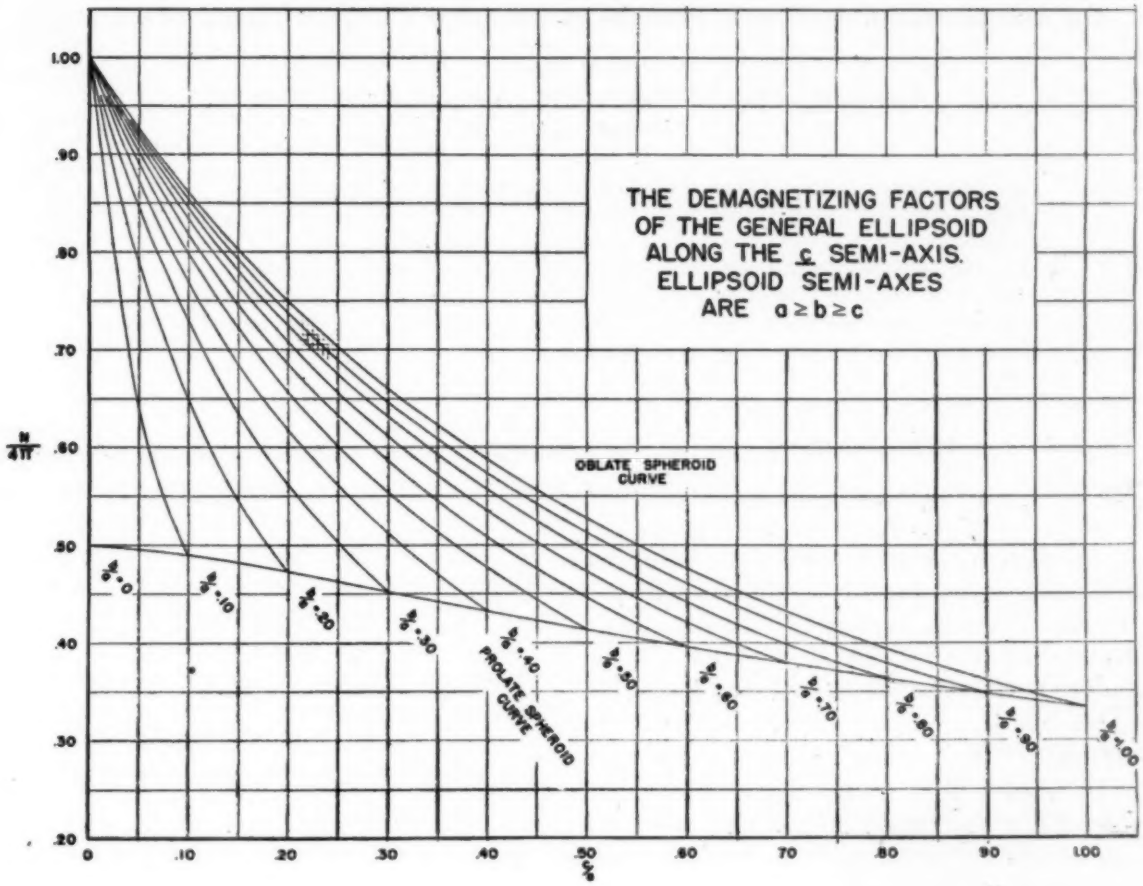

FIG. 3.

μ_i, corresponding to axis i, are constant. (For isotropic materials the three χ_i's are equal.) Therefore

$$H_i=(H_0)_i-N_iJ_i=(H_0)_i-N_i\chi_iH_i, \quad (4.2)$$

so that

$$H_i=(H_0)_i/(1+N_i\chi_i), \quad (4.3)$$

and

$$J_i=\chi_iH_i=\chi_i(H_0)_i/(1+N_i\chi_i)$$
$$=(H_0)_i/(N_i+1/\chi_i). \quad (4.4)$$

By use of these formulas, the magnitude and direction of the magnetizing force and magnetization may be found if the magnitude and direction of the applied field are given.

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On Accelerated Coordinate Systems in Classical and Relativistic Mechanics

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A simplified discussion is given of the problem of introducing linearly accelerated axes by a method applicable alike to classical and relativistic mechanics. The mathematical method is based on the Lie theory of the 4-dimensional conformal group which is presented in such a way that the transformations leading to accelerated axes are exhibited explicitly. The classical theory is shown to be a degenerate case of the relativistic theory, so that the transformations to accelerated axes have a geometrical interpretation in terms of inversive transformations in both theories.

1. INTRODUCTION

FROM time to time, modifications of the special theory of relativity to include transformations between relatively accelerated Euclidean coordinate systems have been discussed in the literature. The first steps in this direction seem to have been taken by Einstein[1] and Born.[2] More recently, the question has been investigated again by Milne[3] in connection with his theory of the expanding universe, while Page[4] has proposed a detailed theory of the kinematical side of the problem. The work of Milne and Page has been subjected to critical discussion by Bourgin,[5] and more particularly by Robertson[6] who has given a very penetrating treatment of the general mathematical theory, with particular reference to its implications in cosmological theories. Finally, Engstrom and Zorn[7] have pointed out independently the connection between Page's theory and the 4-dimensional conformal transformation group. Unfortunately, the widely divergent physical and mathematical viewpoints of the various authors make it difficult to arrive at a clear view of the physical implications of their discussions.

The present work originated in an independent attempt to develop a relativistic equivalent of the simple process in Newtonian mechanics whereby one introduces a system of coordinates moving with an accelerated particle. In this paper we shall give a direct solution of this problem by the simplest applicable means and in such a manner as to bring out the complete analogy between the classical and relativistic theories.

While the present discussion is non-quantum mechanical in character, it may be noted that the establishment of wave equations invariant under the conformal group has been discussed in the literature.[8]

2. STATEMENT OF THE PROBLEM

The starting point of our discussion will be the replacement of the ultimate problem of establishing transformations between accelerated coordinate systems by another which at first sight may perhaps appear to have only a trivial significance. However, its solution gives the key to the more general question.

Apart from the trivial case of uniform motion under no forces, the simplest situation in mechanical theory for which a clear-cut formulation can be given without involving difficulties in the discussion of force fields is that of the uniformly accelerated motion of a particle. This can be characterized in both classical and relativistic theories as a motion in which the acceleration is constant when measured in a system of coordinates in which the particle is instantaneously at rest. Such a set of coordinates is called a *rest-system;* a set of coordinates in which the particle is not only at rest but also has no acceleration will be designated as a *proper-system.*

[1] A. Einstein, Jahrbuch der Radioaktivität **4**, 411 (1907).
[2] M. Born, Ann. d. Physik **30**, 1 (1909). A brief discussion is also given by H. Bateman, Proc. Lond. Math. Soc. [2] **8**, 223 (1910).
[3] E. A. Milne, *Relativity, Gravitation and World-Structure* (Oxford University Press, New York, 1935).
[4] L. Page, Phys. Rev. **49**, 254 (1936).
[5] D. Bourgin, Phys. Rev. **50**, 864 (1936).
[6] H. P. Robertson, Phys. Rev. **49**, 755 (1936); Zeits. f. Astrophys. **7**, 153 (1933); Astrophys. J. **82**, 284 (1935), **83**, 187 (1936), **83**, 257 (1936).
[7] H. T. Engstrom and M. Zorn, Phys. Rev. **49**, 701 (1936).
[8] P. A. M. Dirac, Ann. Math. [2] **37**, 429 (1936); H. J. Bhabha, Proc. Camb. Phil. Soc. **32**, 622 (1936).

Since a particle at rest is but a special case of one in uniformly accelerated motion, the transformation from a rest-system to a proper-system has the property that it transforms one type of uniformly accelerated motion into another. This suggests the following formulation of the problem: *To determine those transformations for which the uniformly accelerated motion of a particle is transformed into another of the same type.*

In this form the problem can be attacked in the same manner in both classical and relativistic theories, and it has the further advantage of suggesting of itself the most direct method of solution. In the discussion of the coordinate transformations which arise, we shall proceed on the consideration that each such set of coordinates may be interpreted as associated with an "observer" to whom the motion of the particle as described by his coordinates is its true motion.

3. THE CLASSICAL ONE-DIMENSIONAL PROBLEM

In order to clarify the essentials of the method, we start with the simplest possible classical case. We consider a particle which in some appropriate initial coordinate system moves along a straight line with a constant acceleration g. Using x as its coordinate, the differential equation of its motion is

$$d^2x/dt^2=g. \tag{1}$$

The differential equation characterizing *all* uniformly accelerated motions is therefore

$$d^3x/dt^3=0.$$

With the notation

$$v=dx/dt, \quad a=dv/dt, \quad b=da/dt,$$

this characteristic differential equation becomes

$$F_c(x, t, v, a, b)\equiv b=0. \tag{2}$$

Our problem is now reduced to that of seeking all transformations of coordinates which leave this equation invariant. In each such new system, the particle will appear to move with constant acceleration, but the magnitude of the acceleration will vary with the system.

A study of the transformation properties of Eq. (2) shows that the only point-transformations which leave it invariant are those generated by the following set of operators:[9]

$$\begin{aligned} &X_1\equiv-\partial_x, && X_2\equiv-\partial_t,\\ &X_3\equiv-t\partial_x, && X_4\equiv-\tfrac{1}{2}t^2\partial_x,\\ &X_5\equiv-xt\partial_x-\tfrac{1}{2}t^2\partial_t, && X_6\equiv x\partial_x+t\partial_t,\\ & && X_7\equiv x\partial_x-t\partial_t, \end{aligned} \tag{3}$$

where, for simplification, $\partial_x\equiv\partial/\partial x$, $\partial_t\equiv\partial/\partial t$.

For convenience we give also the formulas for the general infinitesimal transformation, extended three times to cover the order of Eq. (2):

$$\begin{aligned} x'&=x-\alpha_1-t\alpha_3-\tfrac{1}{2}t^2\alpha_4-xt\alpha_5+x\alpha_6+x\alpha_7,\\ t'&=t-\alpha_2-\tfrac{1}{2}t^2\alpha_5+t\alpha_6-t\alpha_7,\\ v'&=v-\alpha_3-t\alpha_4-x\alpha_5+2v\alpha_7,\\ a'&=a-\alpha_4-(v-ta)\alpha_5-a\alpha_6+3a\alpha_7,\\ b'&=b+2tb\alpha_5-2b\alpha_6+4b\alpha_7. \end{aligned} \tag{4}$$

Since these are extended point-transformations, the formulas for v, a, and b are consequences of those for x and t.

The invariance of Eq. (2) follows at once from the fact that the quantity $(b'-b)$ vanishes with b; i.e., vanishes when Eq. (2) is satisfied.

The requirements of classical theory now lead us to anticipate the possibility of finding a 4-parameter subgroup of transformations which will include one of the type sought to an accelerated coordinate system. The four parameters would be used, (a) to permit shifting the origin in the (x, t)-plane in order to give the particle the position $x=0$ at a given instant $t=0$ (2 parameters), (b) to transform to a rest-system with the particle at the origin (1 parameter), and finally, (c) to transform to the proper-system with the particle still at the origin (1 parameter). It is to be expected also that this can be achieved using only transformations for which time intervals are invariant.

A study of Eqs. (4) shows that this can just be accomplished by using the four transformations generated by X_1, X_2, X_3, X_4. It follows from the results to be obtained in the next section that these transformations form a 4-parameter group. The finite transformations of this group can be

[9] Throughout the paper the operators have been written in such a manner as to facilitate the physical interpretation of the resulting transformations.

calculated readily[10] and reduced to the form

$$x'=(x-x_0)-v_0(t-t_0)-\tfrac{1}{2}a_0(t-t_0)^2, \quad t'=t-t_0,$$

with the notation

$$x_0=\alpha_1+(3\alpha_2\alpha_3+\alpha_4\alpha_2^2)/6; \quad t_0=\alpha_2,$$
$$v_0=\alpha_3+\alpha_2\alpha_4/2; \quad a_0=\alpha_4.$$

In this form it is immediately recognizable as a change to moving, accelerated coordinates with superposed changes of origin for x and t. By a proper choice of x_0 and t_0, we bring the particle to the origin in the (x, t)-plane; by equating v_0 to the velocity of the particle, we obtain a rest-system; and by equating a_0 to the acceleration of the particle in the rest-system, we reduce it to a proper-system. For general values of a_0, we obtain a 1-parameter family of accelerated axis systems.

4. THE RELATIVISTIC ONE-DIMENSIONAL PROBLEM

In the special theory of relativity, the equation of uniformly accelerated motion, replacing Eq. (1), is

$$c^3a/(c^2-v^2)^{\frac{3}{2}}=g, \tag{5}$$

where g is the constant acceleration as measured in a rest-system. By a further differentiation we find the differential equation characteristic of all uniformly accelerated motions to be

$$F_r(x, t, v, a, b)\equiv b+3va^2/(c^2-v^2)=0. \tag{6}$$

The point-transformations which leave this equation invariant form a 6-parameter group built on the operators:[11]

$$\begin{aligned}
&X_1\equiv-\partial_x, \quad X_2\equiv-\partial_t,\\
&X_3\equiv-t\cdot\partial_x-(x/c^2)\cdot\partial_t,\\
&X_4\equiv-(c^2t^2+x^2)/2c^2\cdot\partial_x-(xt/c^2)\cdot\partial_t,\\
&X_5\equiv-xt\cdot\partial_x-(c^2t^2+x^2)/2c^2\cdot\partial_t,\\
&X_6\equiv x\partial_x+t\partial_t.
\end{aligned} \tag{7}$$

[10] The necessary theory of continuous groups can be found in the book of J. E. Campbell, *Theory of Continuous Groups* (Oxford University Press, New York, 1903) or that of G. Kowalewski, *Einführung in die Theorie der Kontinuierlichen Gruppen* (Akademische Verlagsgesellschaft, Leipzig, 1931). The method of finding the finite transformations of a group which is defined in terms of its infinitesimal elements is discussed in Campbell, p. 47.

[11] The full set of symmetry transformations of both Eqs. (2) and (6) form a 10-parameter family, of which the remaining ones are contact transformations, but not point-transformations.

TABLE I. Commutator table of the symmetry operators of relativistic one-dimensional uniformly accelerated motion.

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0					
X_2	0	0				
X_3	X_2/c^2	X_1	0			
X_4	$-X_6/c^2$	X_3	X_5/c^2	0		
X_5	X_3	$-X_6$	X_4	0	0	
X_6	$-X_1$	$-X_2$	0	X_4	X_5	0

For convenience of reference we tabulate the general infinitesimal transformations to the third orders of derivatives:

$$\begin{aligned}
x'&=x-\alpha_1-\alpha_3t-\alpha_4(c^2t^2+x^2)/2c^2-\alpha_5xt+\alpha_6x,\\
t'&=t-\alpha_2-\alpha_3x/c^2-\alpha_4xt/c^2\\
&\qquad-\alpha_5(c^2t^2+x^2)/2c^2+\alpha_6t,\\
v'&=v-\alpha_3(c^2-v^2)/c^2-\alpha_4t(c^2-v^2)/c^2\\
&\qquad-\alpha_5x(c^2-v^2)/c^2,\\
a'&=a+\alpha_3 3va/c^2+\alpha_4(xa+3tva+v^2-c^2)/c^2\\
&\qquad+\alpha_5[c^2ta+3xva-c^2v+v^3]/c^2-\alpha_6a,\\
b'&=b+\alpha_3(4vb+3a^2)/c^2\\
&\quad+\alpha_4(2xb+4tvb+6va+3ta^2)/c^2\\
&\quad+\alpha_5(2c^2tb+4xvb+6v^2a+3xa^2)/c^2-\alpha_6 2b.
\end{aligned} \tag{8}$$

It is readily verified that this transformation leaves Eq. (6) invariant.

Table I is the commutator table for the set of operators of Eq. (7).

Each entry in Table I gives the commutator of the operator at the side of its row with that at the bottom of its column. The operators X_1, X_2, X_3 generate the 3-parameter subgroup of translation and velocity transformations, while X_4, X_5, X_6 generate a second 3-parameter subgroup. Neither of these subgroups is normal within the whole group.

In order to obtain the operators of Section 3, we need only take the limit $c\to\infty$ in the operators

of Eq. (7), in the general infinitesimal transformations of Eq. (8), and in Table I.[12]

To develop the "acceleration transformations" we now suppose that the starting coordinate system has been chosen as a rest-system in which the particle is brought to the origin in the (x, t)-plane. Equations (8) show that the subgroup of transformations generated by X_4, X_5, X_6 has the characteristic that the particle is left at the origin at rest.

The 1-parameter subgroup of transformations generated by X_4 alone gives a family in which the particle is left at the origin at rest, but for which its acceleration becomes $(g-\alpha_4)$. By equating α_4 to g, the proper-system of the particle is found. The finite transformations of this family are easily computed and found to give exactly the result found by Page[4] in his Eqs. (27) and (28), with $\alpha_4=\phi$.[13]

5. GEOMETRICAL INTERPRETATION

Each of the transformations used can be given a geometrical interpretation as an operation in the (x, t)-plane: (a) X_1 and X_2 generate translations of the origin, (b) X_3 has the usual interpretation given in the special theory of relativity as a real rotation in the complex (x, ict)-plane or as an imaginary rotation in the real plane, (c) in the real plane we define the transformation

$$T(x_0, t_0): \quad x'-x_0=\frac{R^2(x-x_0)}{(x-x_0)^2-c^2(t-t_0)^2};$$

$$t'-t_0=\frac{R^2(t-t_0)}{(x-x_0)^2-c^2(t-t_0)^2},$$

as an inversion in the hyperbola

$$(x-x_0)^2-c^2(t-t_0)^2=R^2.$$

If we let $S(x_0, t_0)$ be a translation of the origin through the distances x_0 and t_0 along the axes, then the infinitesimal transformations α_4X_4 and α_5X_5 are given by the following sequences

$$\alpha_4X_4: \quad S(-R^2\alpha_4/2c^2, 0)\cdot T(-R^2\alpha_4/2c^2, 0)\cdot T(0, 0),$$

$$\alpha_5X_5: \quad S(0, R^2\alpha_5/2c^2)\cdot T(0, R^2\alpha_5/2c^2)\cdot T(0, 0),$$

the order of the transformations being read from right to left. In the complex plane, these become combinations of inversions in circles combined with translations,[14] and (d) the transformations generated by X_6 are uniform stretchings and contractions of the (x, t)-plane.

From this we see that, even in the classical theory, the transformations to accelerated coordinate systems have a geometrical interpretation in terms of inversions. This result carries over unchanged into the 3-dimensional theory.

6. THREE-DIMENSIONAL MOTION

The characteristic differential equation for 3-dimensional uniformly accelerated motion is so complex that a direct study of its invariance would be very laborious. It is not difficult to set up the generalization of the operator system of Section 4 by direct evaluation, but it can be put on a more elegant basis by recognizing at once the connection with Lie's theory of the group of conformal transformations in 4 dimensions. The physical basis of the connection is that since a particle traveling with the velocity of light represents a special case of uniform acceleration in which further acceleration is impossible, the family of paths of light rays must therefore be invariant under the transformations. Since the differential equation of a light surface in the starting system is

$$(dx)^2+(dy)^2+(dz)^2-c^2(dt)^2=0,$$

this relation must also be satisfied after the transformation, which implies that

$$(dr')^2-c^2(dt')^2=\rho[(dr)^2-c^2(dt)^2],$$

[12] The operator X_7 of Section 3 appears in the relativistic theory as a degenerate case of a contact transformation which becomes a point-transformation on taking the limit $c\to\infty$. The group used here is isomorphic to the symmetry group of the circles in the complex (x, ict)-plane. The passage from the relativistic to the classical theory appears in Lie's geometrical theory as the transition from the symmetry group of circles in the complex plane to that of the family of parabolas in the real plane forming the classical trajectories of possible uniformly accelerated particles. This geometrical theory is discussed in the books of Campbell and of Kowalewski mentioned in reference 10.

[13] The finite transformations of the 2-parameter subgroup based on X_4 and X_5 have been computed. The equations show three singular lines, of which two have the equations

$$(x\pm ct)(\alpha_4\pm\alpha_5c)+2c^2=0$$

and are the generalizations of those noted by Page. The third is the line

$$\alpha_5x-\alpha_4t=0$$

which transforms into itself according to the relation

$$(\alpha_5x'-\alpha_4t')\cdot(x^2-c^2t^2)=(\alpha_5x-\alpha_4t)\cdot(x'^2-c^2t'^2).$$

[14] A discussion of this 2-dimensional geometrical theory can be found in Chap. 19 of Campbell, reference 10.

TABLE II. Commutator table of the conformal group.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}
X_1	0														
X_2	0	0													
X_3	0	0	0												
X_4	0	0	0	0											
X_5	0	X_3	$-X_2$	0	0										
X_6	$-X_3$	0	X_1	0	$-X_7$	0									
X_7	X_2	$-X_1$	0	0	X_6	$-X_5$	0								
X_8	X_4/c^2	0	0	X_1	0	X_{10}	$-X_9$	0							
X_9	0	X_4/c^2	0	X_2	$-X_{10}$	0	X_8	X_7/c^2	0						
X_{10}	0	0	X_4/c^2	X_3	X_9	$-X_8$	0	$-X_6/c^2$	X_5/c^2	0					
X_{11}	$-X_{13}/c^2$	X_7/c^2	$-X_6/c^2$	X_8	0	X_{13}	$-X_{12}$	X_{14}/c^2	0	0	0				
X_{12}	$-X_7/c^2$	$-X_{13}/c^2$	X_5/c^2	X_9	$-X_{13}$	0	X_{11}	0	X_{14}/c^2	0	0	0			
X_{13}	X_6/c^2	$-X_5/c^2$	$-X_{12}/c^2$	X_{10}	X_{12}	$-X_{11}$	0	0	0	X_{14}/c^2	0	0	0		
X_{14}	X_8	X_9	X_{10}	$-X_{15}$	0	0	0	X_{11}	X_{12}	X_{13}	0	0	0	0	
X_{15}	$-X_1$	$-X_2$	$-X_3$	$-X_4$	0	0	0	0	0	0	X_{11}	X_{12}	X_{13}	X_{14}	0

where ρ is an unspecified function of x, y, z, t. This relation may be used to characterize the group of transformations.[7,15]

We proceed at once to give the set of 15 operators which form the basis of the group of transformations:

$$X_1 \equiv -\partial_x, \quad X_2 \equiv -\partial_y, \quad X_3 \equiv -\partial_z, \quad X_4 \equiv -\partial_t,$$

$$X_5 \equiv -y\partial_z + z\partial_y, \quad X_6 \equiv -z\partial_x + x\partial_z,$$

$$X_7 \equiv -x\partial_y + y\partial_x, \quad X_8 \equiv -t\partial_x - (x/c^2)\partial_t,$$

$$X_9 \equiv -t\partial_y - (y/c^2)\partial_t, \quad X_{10} \equiv -t\partial_z - (z/c^2)\partial_t,$$

$$X_{11} \equiv -(c^2t^2+x^2-y^2-z^2)/2c^2\cdot\partial_x - (xy/c^2)\partial_y - (xz/c^2)\partial_z - (xt/c^2)\partial_t,$$

$$X_{12} \equiv -(xy/c^2)\partial_x - (c^2t^2-x^2+y^2-z^2)/2c^2\cdot\partial_y - (yz/c^2)\partial_z - (yt/c^2)\partial_t,$$

$$X_{13} \equiv -(xz/c^2)\partial_x - (yz/c^2)\partial_y - (c^2t^2-x^2-y^2+z^2)/2c^2\cdot\partial_z - (zt/c^2)\partial_t,$$

$$X_{14} \equiv -xt\partial_x - yt\partial_y - zt\partial_z - (c^2t^2+x^2+y^2+z^2)/2c^2\cdot\partial_t,$$

$$X_{15} \equiv x\partial_x + y\partial_y + z\partial_z + t\partial_t.$$

The commutator relations of these operators are given in Table II.

X_1, X_2, X_3, X_4 generate the translations. X_5, X_6, X_7, generate the 3-dimensional rotations, while X_8, X_9, X_{10} generate the Lorentz transformations. X_{11}, X_{12}, X_{13} form the 3-dimensional acceleration sub-set, of which X_{14} gives the time-component. X_{15} again gives the homogeneous dilatations of the whole space.

The procedure for studying the transformation to accelerated axes proceeds by the same principles as in Section 4. By extending the transformations as far as the acceleration terms, one finds that if the particle is brought to the origin of coordinates in a rest-system, then it will be

[15] S. Lie, *Theorie der Transformationsgruppen* (Teubner, Leipzig, 1930). A very interesting, but difficult, discussion is given in a classic paper by H. Bateman, Proc. Lond. Math. Soc. [2] **8**, 223 (1910).

left in this condition under all transformations of the acceleration subgroup; i.e., under the 3-parameter family of transformations generated by X_{11}, X_{12}, X_{13}.[16]

On orienting the x-axis along the direction of the acceleration of the particle, the 1-parameter family of transformations which include its proper-system can be found. The transformation equations are:

$$x'=[x+\alpha_{11}(r^2-c^2t^2)/2c^2]\cdot(t'/t),$$

$$y'=y\cdot(t'/t),\quad z'=z\cdot(t'/t),$$

$$t'=t/\{1+(\alpha_{11}/c^2)[x+(\alpha_{11}/4c^2)(r^2-c^2t^2)]\},$$

$$r^2=x^2+y^2+z^2.$$

The passage to the classical theory is readily made, as in Section 4, merely by taking the limit $c\to\infty$ in all of the equations for the operators as well as in the commutator table. The general infinitesimal transformation may then be written in 3-dimensional vector form as

$$\delta\mathbf{r}\equiv\mathbf{r}'-\mathbf{r}=-\delta\mathbf{s}-\delta\boldsymbol{\omega}\times\mathbf{r}-t\delta\mathbf{v}-\tfrac{1}{2}t^2\delta\mathbf{a}-\mathbf{r}t\alpha_{14}+\mathbf{r}\alpha_{15},$$

$$\delta t\equiv t'-t=-\alpha_4-\tfrac{1}{2}t^2\alpha_{14}+t\alpha_{15},$$

with the abbreviations

$$\delta\mathbf{s}=(\alpha_1\mathbf{i}+\alpha_2\mathbf{j}+\alpha_3\mathbf{k}),$$

$$\delta\boldsymbol{\omega}=(\alpha_5\mathbf{i}+\alpha_6\mathbf{j}+\alpha_7\mathbf{k}),$$

$$\delta\mathbf{v}=(\alpha_8\mathbf{i}+\alpha_9\mathbf{j}+\alpha_{10}\mathbf{k}),$$

$$\delta\mathbf{a}=(\alpha_{11}\mathbf{i}+\alpha_{12}\mathbf{j}+\alpha_{13}\mathbf{k}).$$

The operators X_{14} and X_{15} can be dropped, and the transformations to accelerated axes based on the remaining 13-parameter group. Time intervals are then invariant.

[16] The particle at the origin in a rest system is left at that position at rest under the 5-parameter group generated by X_{11}, X_{12}, X_{13}, X_{14}, X_{15}.

7. ELECTRODYNAMIC CONSIDERATIONS

The detailed proof of the invariance of the electromagnetic field equations and the derivation of the transformation relations of the field vectors and potentials have been given by Cunningham[17] and Bateman.[15] However, this only assures us that electromagnetic phenomena as observed by any of the family of observers related by transformations of the conformal group will appear to obey the field equations. Since the transformations are in general non-linear, plane waves do not transform into plane waves, and the existence of singular regions in the finite transformations makes it unclear to what extent retarded and advanced potentials may be intermixed.

We shall note here in conclusion that if one assumes that in its proper-system a point charge exhibits its usual Coulomb field, then Bateman's formulas can be used to find its field in other sets of axes. A first-order calculation of the field due to a charge e at the origin of coordinates and moving with infinitesimal velocity $\delta\mathbf{v}$ and acceleration $\delta\mathbf{a}$ yields the result

$$\mathbf{E}=e\mathbf{r}/r^3-e(\mathbf{r}\cdot\delta\mathbf{a})\mathbf{r}/2r^3c^2-e\delta\mathbf{a}/2rc^2,$$

$$\mathbf{H}=-e(\mathbf{r}\times\delta\mathbf{v})/r^3c.$$

which is in agreement with the usual formula to this order of approximation.

It is of interest to note also that if the potentials are computed, they contain contributions in α_{14} and α_{15}, even to first orders, but on passing to the field vectors these terms disappear, so that they introduce a type of gauge transformation.

[17] E. Cunningham, Proc. Lond. Math. Soc. [2] 8, 77 (1910).

Letters to the Editor

PROMPT publication of brief reports of important discoveries in physics may be secured by addressing them to this department. The closing date for this department is the third of the month. Because of the late closing date for the section no proof can be shown to authors. The Board of Editors does not hold itself responsible for the opinions expressed by the correspondents. Communications should not in general exceed 600 words in length.

The Magnetic Field Inside a Ferromagnet

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April 13, 1945

IT is generally agreed that for a fast charged particle (in the following, referred to as the "test charge") traversing a magnetized iron bar, the average magnetic field $\mathbf{b}$ is equal to the induction $\mathbf{B}$.[1,2] However, doubts as to the correctness of this statement have been expressed on classical grounds[3] because the dominating contribution to this average is made by the enormous field which exists inside the spinning electron. This same state of affairs prevails in quantum theory. It expresses itself in the formula

$$\mathbf{b}=\mathbf{h}+4\pi p\mathbf{M}. \tag{1}$$

Here $\mathbf{b}$ is the average field, $\mathbf{M}$ is the magnetization, and $\mathbf{h}$ is the field which would exist if the electrons were true magnetic "di-poles." p is a numerical factor which is equal to the relative probability of coincidence of test charge and electron, as compared to randomness.

The formula can be derived without specifying the structure of the wave function, except for the following assumptions:

(a) that the magnetic interaction is sufficiently small to be treated as a first-order perturbation,

(b) that the magnetic field is owing entirely to the electronic spin,

(c) that the ferromagnetic electrons move in orbits independent of each other,

(d) that it is sufficiently accurate to solve the Dirac equation in the Schroedinger type approximation of Darwin-Pauli, and

(e) that the test charge is much heavier than an electron.

Formula (1) becomes standard when the coincidence probability p equals 1, that is for plane waves. However, the term $4\pi\mathbf{M}$ is owing entirely to head-on collisions. This gives rise to two objections:[3]

(1) Head-on collisions may not be frequent enough to average out for each particle. This objection is actually unfounded as such a collision occurs in a crystal about once every hundred atoms.

(2) The coincidence probability p of the test charge and electron may be different from unity.

The latter proposition is certainly true to some extent, partly because of the crystalline field. But the primary reason is the direct attraction or repulsion between electron and test charge upon close approach. Barring the case of uncharged particles such as neutrons, or the existence of short range forces yet unknown, this direct interaction is primarily the Coulomb interaction between electric charges. For this case, the numerical value of p is known.[4] It equals

$$p=x/(1-e^{-x}), \tag{2}$$

where

$$x=4\pi^2Ze^2/hv=\pm 1.39\times 10^9|Z|/v. \tag{3}$$

Here v is the velocity of the test charge in cm/sec. and Ze is the charge of the test charge. The sign of x depends on the sign of this charge. A few values of p are listed in Table I.

TABLE I.

	p	
$v/\|Z\|$	+charge	−charge
$\frac{1}{4}\cdot 10^9$	5.58	0.021
$\frac{1}{2}\cdot 10^9$	2.97	0.184
$1\cdot 10^9$	1.85	0.461
$2\cdot 10^9$	1.39	0.692
$4\cdot 10^9$	1.18	0.837

Formula (1) suggests that if $\mathbf{b}$ is different from $\mathbf{B}$, or $\mathbf{h}$ different from $\mathbf{H}$, it will be primarily caused by short range Coulomb interaction between the test charge and the ferromagnetic electron. Such a calculation can be handled as a collision problem for two particles which are free otherwise. The necessary integrations can then be performed exactly and give

$$\mathbf{b}=\mathbf{H}+2\pi(p+1)\mathbf{M}, \tag{4a}$$

or

$$\mathbf{b}=\mathbf{B}+2\pi(p-1)\mathbf{M}, \tag{4b}$$

where p is again the parameter used in (1) and (2). This gives a field larger than $\mathbf{B}$ for positive charges and smaller than $\mathbf{B}$ for negative charges.

From an experimental point of view, the most hopeful feature for a check is the assymmetry of the equations for positive and negative charges. Positive particles should be deflected more under otherwise similar circumstances, provided their speed is not much greater than 5×10^9 cm/sec. It should be added in conclusion that because of assumption (d), all formulas are subject to relativistic modifications which are being investigated by the author.

[1] C. F. v. Weizsäcker, Ann. d. Physik **17**, 869 (1933).
[2] F. Rasetti, Phys. Rev. **66**, 1 (1944).
[3] W. F. G. Swann, Phys. Rev. **49**, 574 (1936).
[4] Mott and Massey, *Theory of Atomic Collisions* (Oxford University Press, New York), p. 36, formula (17).

Development of Electromagnetic Theory for Non-Homogeneous Spaces—A Correction

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New York, New York
May 5, 1945

MY paper under the above title[1] purported to effect a separation of the components which define an electromagnetic field for the general case in which ϵ, the specific inductive capacity of the space, is an arbitrary twice-differentiable function of the coordinates. The

method consists in introducing a Riemannian line element (Eq. (2) of the paper) with $h^2=1/\epsilon$, and also in introducing an auxiliary scalar function Ω for the purpose of obtaining the desired generality. This function Ω is related to the h of the line element by Eq. (11). It has been called to my attention, however, that terms in the second derivatives of Ω were omitted in my Eq. (7); these terms cancel the Ω-term in Eqs. (11) so that that function is *not* determined. When the term in Ω is dropped in Eqs. (11), the theory remains valid only if the h of the line element is restricted to the form of a product of functions, *viz.*

$$h=H_1(u_1)H_2(u_2)H_3(u_3), \qquad (A)$$

where the H's are arbitrary functions of their respective arguments with required continuity. Thus, equations such as (12), (18), (20), and (22) remain valid if Ω is put equal to unity and h is restricted in accordance with Eq. (A) above. However, wave equations in which h is put equal to a function of r ($r^2=u_1^2+u_2^2+u_3^2$) are valid only for the very special case in which the function is $\exp kr^2$. All my discussion based on putting h equal to an arbitrary function of r is no longer justified.

Although Eq. (A) still allows considerable latitude in the choice of h, nevertheless, the restriction it imposes is too severe from the standpoint of atomic theory, for which the work was originally intended. Attempts to restore greater generality by introducing a broader line element, *viz.*,

$$ds^2=(f_1^2du_1^2+f_2^2du_2^2+f_3^2du_3^2)/\epsilon,$$

do not appear to help the situation materially, because, in order to effect the separation, it appears necessary to restrict the coefficients so that

$$f_i\equiv f_i(u_i), \quad \epsilon=g_1(u_1)g_2(u_2)g_3(u_3),$$

where each g and each f is an arbitrary function of the indicated coordinate only. I hope, however, to be able to report better success in the special case of a spherical coordinate system with ϵ a function of r only.

I am greatly indebted to Dr. T. Pearcey of England for calling this error to my attention.

[1]Phys. Rev. **64**, 294 (1943).

On the Evidence for the Existence of Neutral Mesotrons

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June 2, 1945

IT has been observed by Groetzinger and others[1] that when 10.5 Mev deuterons strike targets of Mn, W, P, Sb, or NaCl two-fold coincidences occur in a counter telescope placed outside the cyclotron at a distance of about four meters from the target in the same line as that of the deuteron beam, even when the cyclotron is shielded by water tanks 120 cm thick and the counter telescope is shielded by lead sheets 19 cm thick. The walls of the counters were such that only 2 Mev electrons could pass through both the counters. From results of investigations on the influence of the thickness of the lead shield and water shield on the number of coincidences observed, they have concluded that the electrons responsible for the observed coincidences are produced by the decay of some neutral particles coming from the targets bombarded by 10.5 Mev deuterons, and these particles are probably low mass neutral mesotrons.

Evidence for the presence of such unstable neutral particles in cosmic rays was found in 1942 in some investigations carried out in this department[2] with an anti-coincidence counter telescope placed first with its axis vertical, and next inclined to the vertical at an angle of 30°. It was observed that some neutral particles, passing through the anticounter without discharging it, produce charged mesotrons in a lead absorber 5 cm thick and these mesotrons produce a three-fold coincidence in a counter telescope. The intensity of such neutral particles was found to be about 1.4 percent of the total mesotron intensity in the laboratory, the effect due to side showers being eliminated completely. The intensity was also found to diminish in the inclined direction almost in the same ratio as that of the charged mesotron. The apparatus used is shown diagrammatically in Fig. 1. The absorption coefficient of these particles in

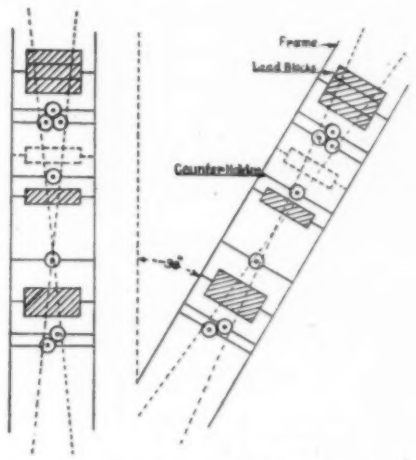

FIG. 1. Arrangement of apparatus in the counter telescope.

lead was found to be less than that for charged mesotrons. From these results it was concluded that these neutral particles are unstable, and they have a life of the same order as that of the charged mesotron. It can be easily seen now that the particles observed by Groetzinger and others to be produced artificially by bombardment of high energy deuterons are identical with those found to be present in cosmic rays. The investigations of Groetzinger and others[1] confirm the inference made from results of cosmic-ray investigations mentioned above that these particles are unstable and give rise to high energy electrons after spontaneous disintegration.

The writers of the note are indebted to Professor M. N. Saha, F.R.S. for kindly drawing their attention to the note by Groetzinger and others.

[1] G. Groetzinger, P. G. Kruger, and Lloyd Smith, Phys. Rev. **67**, 52 (1945).
[2] S. C. Sirkar and P. K. Bhattacharyya, Science and Culture **7**, 89, (1942); Proc. Nat. Inst. Sc. India **10**, 175 (1944).

On Isotopic Spin

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June 2, 1945

SUPPOSE we have a dynamical system containing N heavy nuclear particles. A complete set of commuting observables for a state of system will be r_α, σ_α, τ_α where σ_α is spin and τ_α is isotopic spin. Thus the state of a dynamical nuclear system is determined by the wave functions $\psi(r_1 \cdots r_N;\ \sigma_1 \cdots \sigma_N;\ \tau_1 \cdots \tau_N)$. For the two-valued spin variables σ_α and τ_γ, we have $q \cdot 2^{2N}$ functions

$$\psi^\xi \sigma_1 \cdots \sigma_N;\ \tau_1 \cdots \tau_N{}^{(r_1 \cdots r_N)}, \tag{1}$$

where q is the number of the independent stationary states determined by the Cartesian forces.

The properties of the stationary states of the dynamical system with the N particles can be expressed by the *irreducible* representations of the symmetrical group of N variables.

The $2^{2N} \cdot q$ dimensional representation D^A with the basis (1) splits under the transformations of the symmetrical group into irreducibly invariant subspaces D_α,

$$D^A = D_\sigma \times D_\tau \times D_\omega = \Sigma_\alpha D_\alpha, \tag{2}$$

each defined by *Partitio Numerorum*

$$N = K_1 + K_2 + \cdots + K_\alpha, \qquad K_1 \leq K_2 \leq \cdots \leq K_\alpha, \tag{3}$$

where K_α are the whole positive numbers.

The reduction of the D_σ (and D_τ) is determined by the following theorem:

$$D^\sigma = \sum_{s,m} D^{s,m}, \quad D^\tau = \sum_{T,\tau} D^{T,\tau}, \tag{4}$$

where $D^{s,m}$, $D^{T,\tau}$ are the irreducible representations and

$$-s \leq m \leq +s, \quad s = (N/2), (N/2)-1, \cdots \tfrac{1}{2} \text{ or } 0,$$
$$-T \leq \tau \leq +T, \quad T = (N/2), (N/2)-1, \cdots \tfrac{1}{2} \text{ or } 0, \tag{5}$$

the number of the rows and columns of the matrices $(D^{s,m})_{\alpha\beta}$, $(D^{T,\tau})_{\rho\sigma}$; i.e., the number of independent states of the basis of the irreducible representation is:

$$\lambda_T,\ \lambda_s = \left[\binom{N}{(N/2)-s} - \binom{N}{(N/2)-s-1}\right]. \tag{6}$$

The *Partitio Numerorum* $(D^{s,m})$ and $(D^{T,\tau})$ are to be[1]

$$N,\ (N-1)+1,\ (N-2)+2,$$
$$\cdots((N/2)+|s|)-((N/2)-|s|). \tag{7}$$

The reduction of the D^A is thus determined by the properties of the irreducible representation of the symmetrical group D^ω (i.e., by the representation of the Cartesian co-ordinates) and by the Pauli principle.

The formulation of this problem for the ρ-valued spin and ν-valued isotopic spin, i.e., for the many-valued electric charges, is very interesting. The physical properties of the dynamical quantum system are determined by the characters X_ν of the irreducible representations D_α of the symmetrical group.

The study of the possible types of the nuclear interaction is made especially easy by consideration of the possible irreducible representation of the symmetric group. The zero-approximation is determined by Hamiltonian H_0 and the representation D_ω. The interaction may be considered as introducing a splitting of the stationary states of the zero-approximation (the interchange effects for a spin and isotopic spin-variables). The energetic relations can be determined by the general Heitler-Dirac[2] formula by means of the characters X_ν.

[1] J. v. Neumann and E. Wigner, Zeits. f. Physik **47**, 203; **49**, 73 (1928).
[2] W. Heitler, Zeits. f. Physik **51**, 805 (1928); Dirac, Proc. Roy. Soc. **123**, 714 (1929); H. Weyl, *Gnippentheorie and Quantenmechanik* (S. Hirzel, Leipzig, 1931), Vol. 2, Chap. V.

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